Published by Institute of Physics Publishing for SISSA/ISAS

Received: *September 10, 2004*
Revised: *November 10, 2004*
Accepted: *November 12, 2004*

Supersymmetric probes on the conifold

Daniel Areán, David E. Crooks and Alfonso V. Ramallo

Departamento de Física de Partículas, Universidad de Santiago de Compostela
E-15782 Santiago de Compostela, Spain
E-mail: arean@fpaxp1.usc.es, dcrooks@fpaxp1.usc.es, alfonso@fpaxp1.usc.es

Abstract: We study the supersymmetric embeddings of different D-brane probes in the $AdS_5 \times T^{1,1}$ geometry. The main tool employed is kappa symmetry and the cases studied include D3-, D5- and D7-branes. We find a family of three-cycles of the $T^{1,1}$ space over which a D3-brane can be wrapped supersymmetrically and we determine the field content of the corresponding gauge theory duals. Supersymmetric configurations of D5-branes wrapping a two-cycle and of spacetime filling D7-branes are also found. The configurations in which the entire $T^{1,1}$ space is wrapped by a D5-brane (baryon vertex) and a D7-brane are also studied. Some other embeddings which break supersymmetry but are nevertheless stable are also determined.

Keywords: D-branes, Brane Dynamics in Gauge Theories, AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.

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1. Introduction

The AdS/CFT correspondence [1] relates large N gauge theory to string theory. In the limit in which the 't Hooft coupling $g_{YM}^2 N$ becomes infinite, one can neglect the massive modes of the string and take type IIB supergravity on the string theory side of the correspondence [2]. It is nevertheless very interesting to understand how Yang-Mills theory encodes the full features of string theory.

A possible way to uncover stringy effects in Yang-Mills theory consists of adding D-branes on the supergravity side and trying to find out the corresponding field theory dual. This approach was pioneered by Witten in ref. [3]. Indeed, in [3] Witten considered a D3-brane wrapped over a topologically non-trivial cycle of the $AdS_5 \times \mathbb{RP}^5$ background and showed that this configuration is dual to certain operators of dimension N of the SO(N) gauge theory, namely the "pfaffians". Another example along the same lines is provided by the so-called giant gravitons, which are rotating branes wrapped over a topologically trivial cycle [4]. These branes are not topologically stable: they are stabilized dynamically by their angular momentum. The corresponding field theory duals have been found in ref. [5].

In this paper we will study D-brane probes in the so-called Klebanov-Witten model [6]. This model is obtained by placing a stack of N D3-branes at the tip of a conifold. The D3-branes warp the conifold metric and the resulting geometry becomes $AdS_5 \times T^{1,1}$, with N units of Ramond-Ramond flux threading the $T^{1,1}$ space. The corresponding dual field theory is a four-dimensional $\mathcal{N} = 1$ superconformal field theory with gauge group SU(N) $\times$ SU(N) coupled to four chiral superfields in the bifundamental representation.

The effect of adding different D-branes to the Klebanov-Witten background has already been studied in several places in the literature. In ref. [7] it was proposed that D3-branes wrapped over three-cycles of $T^{1,1}$ are dual to dibaryon operators built out of products of N chiral superfields (see also refs. [8]–[11] for more results on dibaryons in this model and in some orbifold theories). Moreover, it was also shown in ref. [7] that a D5-brane wrapped over a two-cycle of $T^{1,1}$ behaves as a domain wall in AdS_5. On the other hand, as first proposed in ref. [12], one can use D7-branes to add dynamical flavor to the Klebanov-Witten model (see also refs. [13]–[16]). A list of the stable D-branes in this background, obtained with methods quite different from those employed here, has appeared in ref. [17].

The main technique that we will employ to determine the supersymmetric embeddings of the different D-brane probes in the $AdS_5 \times T^{1,1}$ background is kappa symmetry [18]. This approach is based on the fact that there exists a matrix Γ_κ such that, if ϵ is a Killing spinor of the background, only those embeddings for which $\Gamma_\kappa \epsilon = \epsilon$ preserve some supersymmetry of the background [19]. The matrix Γ_κ depends on the metric induced on the worldvolume of the probe and, therefore, if the Killing spinors are known, the kappa symmetry condition gives rise to a set of first-order differential equations whose solutions (if they exist) determine the supersymmetric embedding of the brane probe. For these configurations the kappa symmetry equation introduces some additional conditions

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on ϵ, which are only satisfied by some subset of the Killing spinors. Thus, the probe only preserves some fraction of the original supersymmetry of the background. For all the solutions we will find here we will be able to identify the supersymmetry that they preserve. Moreover, we will verify that the corresponding embeddings satisfy the equations of motion derived from the Dirac-Born-Infeld action of the probe. Actually, in all the cases studied, we will establish a series of BPS bounds for the energy, along the lines of those studied in ref. [20], which are saturated by the kappa symmetric embeddings.

Clearly, to carry out the program sketched above we need to have a detailed knowledge of the Killing spinors of the $AdS_5 \times T^{1,1}$ background. In particular, it would be very useful to find a basis of frame one-forms in which the spinors do not depend on the coordinates of the $T^{1,1}$ space. It turns out that this frame is provided very naturally when the conifold is obtained [21] from an uplifting of eight-dimensional gauged supergravity [22]. In this frame the Killing spinors are characterized by simple algebraic conditions, and one can systematically explore the solutions of the kappa symmetry equation.

The first case we will study is that of the supersymmetric embeddings of D3-brane probes. We will find a general family of three-cycles which contains, as a particular case, the one used in ref. [7] to describe the dual of the dibaryonic operator. We will be able to identify the field theory content of the operators dual to our D3-brane embeddings. Moreover, we will also find two-cycles on which the D3-brane can be wrapped in such a way that the equations of motion are satisfied and, despite of the fact that supersymmetry is completely broken, the system is stable.

We will consider next D5-brane probes, for which we will be able to identify the two-cycle on which the D5-brane must be wrapped to realize the domain wall of the four-dimensional gauge theory. We will also verify that if we wrap the D5-brane over the same three-cycles which made the D3-brane supersymmetric, one gets a non-supersymmetric stable solution of the equations of motion of the D5-brane probe. The baryon vertex for the Klebanov-Witten model, a D5-brane wrapped over the entire $T^{1,1}$, will be also analyzed. We will argue that this configuration cannot be supersymmetric.

Our final case is that corresponding to D7-brane probes. We will first study the spacetime filling configurations. In this case we will be able to find a two-parameter family of supersymmetric embeddings which, in particular, include those proposed in refs. [12, 14] as suitable to add flavor to this background. Our results confirm that these configurations are kappa symmetric. We will also show that the D7-brane can wrap the entire $T^{1,1}$ and preserve some supersymmetry.

This paper is organized as follows. In section 2 we review the basic features of the Klebanov-Witten model. In particular, we give the explicit form of the Killing spinors in the frame which is more adequate to our purposes. We also introduce in this section the general form of the kappa symmetry matrix Γ_κ and discuss the general strategy to solve the $\Gamma_\kappa \epsilon = \epsilon$ equation.

Section 3 is devoted to the analysis of the supersymmetric D3-brane embeddings. After choosing a set of convenient worldvolume coordinates and an ansatz for the scalar fields that determine the embedding, we will be able to find a pair of first-order differential equations whose solutions determine the supersymmetric wrappings of the D3-brane over

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a three-cycle. This pair of equations can be solved in general after a change of variables which converts them into the Cauchy-Riemann equations. Similar analysis are carried out for the D5-brane and D7-brane probes in sections 4 and 5 respectively. Some other possible embeddings for D3-, D5- and D7-branes are discussed in appendix A. In section 6 we summarize our results and draw some conclusions. We have also included in appendix B the calculation of the Killing spinors of the so-called Klebanov-Strassler background [23], which could serve as a starting point to generalize our results to backgrounds dual to theories without conformal invariance.

2. The Klebanov-Witten model

The conifold is a non-compact Calabi-Yau threefold with a conical singularity. Its metric can be written as $ds_6^2 = dr^2 + r^2\, ds^2_{T^{1,1}}$, where $ds^2_{T^{1,1}}$ is the metric of the $T^{1,1}$ coset $(\mathrm{SU}(2)\times \mathrm{SU}(2))/\,\mathrm{U}(1)$, which is the base of the cone. The $T^{1,1}$ space is an Einstein manifold whose metric can be written [24] explicitly by using the fact that it is an U(1) bundle over $S^2\times S^2$. Actually, if (θ_1,ϕ_1) and (θ_2,ϕ_2) are the standard coordinates of the S^2's and if $\psi\in[0,4\pi)$ parametrizes the U(1) fiber, the metric may be written as:

$$ds^2_{T^{1,1}} = \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i\, d\phi_i^2 \right) + \frac{1}{9}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i \right)^2 . \tag{2.1}$$

The conifold can also be described as the locus of points in $\mathbb{C}^4$ which satisfy the equation:

$$z_1\, z_2 - z_3\, z_4 = 0\,, \tag{2.2}$$

which obviously has an isolated conical singularity at the origin of $\mathbb{C}^4$. The relation between the holomorphic coordinates z_i with the angles and r is:

$$\begin{aligned}
z_1 &= r^{3/2}\, e^{\frac{i}{2}(\psi-\phi_1-\phi_2)}\,\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}\,, \qquad & z_2 &= r^{3/2}\, e^{\frac{i}{2}(\psi+\phi_1+\phi_2)}\,\cos\frac{\theta_1}{2}\,\cos\frac{\theta_2}{2}\,,\\
z_3 &= r^{3/2}\, e^{\frac{i}{2}(\psi+\phi_1-\phi_2)}\,\cos\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2}\,, \qquad & z_4 &= r^{3/2}\, e^{\frac{i}{2}(\psi-\phi_1+\phi_2)}\,\sin\frac{\theta_1}{2}\,\cos\frac{\theta_2}{2}\,.
\end{aligned} \tag{2.3}$$

It is also interesting to find some combinations of the z_i's which only depend on the coordinates (θ_1,ϕ_1) or (θ_2,ϕ_2). Actually, from the parametrization (2.3) it is straightforward to prove that:

$$\frac{z_1}{z_3} = \frac{z_4}{z_2} = e^{-i\phi_1}\,\tan\frac{\theta_1}{2}\,, \qquad \frac{z_1}{z_4} = \frac{z_3}{z_2} = e^{-i\phi_2}\,\tan\frac{\theta_2}{2}\,. \tag{2.4}$$

By adding four Minkowski coordinates to the conifold we obtain a Ricci flat ten-dimensional metric. Let us now place in this geometry a stack of N coincident D3-branes extended along the Minkowski coordinates and located at the singular point of the conifold. The resulting model is the so-called Klebanov-Witten (KW) model. The corresponding

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near-horizon metric and Ramond-Ramond selfdual five-form are given by:

$$\begin{aligned}
ds^2_{10} &= [h(r)]^{-1/2}\, dx^2_{1,3} + [h(r)]^{1/2}\, (\, dr^2 + r^2\, ds^2_{T^{1,1}}\,)\,,\\
h(r) &= \frac{L^4}{r^4}\,,\\
g_s\, F^{(5)} &= d^4x\, \wedge dh^{-1} + \text{Hodge dual}\,,\\
L^4 &= \frac{27}{4}\,\pi g_s N \alpha'^2\,.
\end{aligned} \tag{2.5}$$

The gauge theory dual to the supergravity background (2.5) is an $\mathcal{N}=1$ superconformal field theory with some matter multiplets. Actually, the metric in (2.5) can be written as:

$$ds^2_{10} = \frac{r^2}{L^2}\, dx^2_{1,3} + \frac{L^2}{r^2}\, dr^2 + L^2\, ds^2_{T^{1,1}}\,, \tag{2.6}$$

which corresponds to the $AdS_5 \times T^{1,1}$ space. In order to exhibit the field theory dual to this background, let us solve the conifold equation (2.2) by introducing four homogeneous coordinates A_1, A_2, B_1 and B_2 as follows:

$$z_1 = A_1\, B_1\,, \qquad z_2 = A_2\, B_2\,, \qquad z_3 = A_1\, B_2\,, \qquad z_4 = A_2\, B_1\,. \tag{2.7}$$

Following the analysis of ref. [6], one can show that the dual superconformal theory can be described as an $\mathcal{N}=1$ $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge theory which includes four $\mathcal{N}=1$ chiral multiplets, which can be identified with (the matrix generalization of) the homogeneous coordinates A_1, A_2, B_1 and B_2. The fields A_1 and A_2 transform in the $(\mathbf{N},\bar{\mathbf{N}})$ representation of the gauge group, while B_1 and B_2 transform in the $(\bar{\mathbf{N}},\mathbf{N})$ representation. These fields are coupled through an exactly marginal superpotential W of the form:

$$W = \lambda\, \epsilon^{ij}\, \epsilon^{kl}\, tr(A_i B_k A_j B_l)\,, \tag{2.8}$$

where λ is a constant. The R-charge of the A and B fields is 1/2, whereas their conformal dimension is 3/4.

2.1 Killing spinors

As argued in ref. [6], the KW model preserves eight supersymmetries (see also ref. [25]). Notice that this is in agreement with the $\mathcal{N}=1$ superconformal character of the corresponding dual field theory, which has four ordinary supersymmetries and four superconformal ones.

To obtain the explicit form of the Killing spinors, one has to look at the supersymmetry variations of the dilatino and gravitino (see eq. (B.1)). It turns out that the final result of the calculation is greatly simplified if some particular basis of the frame one-forms for the $T^{1,1}$ part of the metric is chosen. In order to specify this basis, let us define three one-forms associated to a two-sphere

$$\sigma^1 = d\theta_1\,, \qquad \sigma^2 = \sin\theta_1\, d\phi_1\,, \qquad \sigma^3 = \cos\theta_1 d\phi_1\,, \tag{2.9}$$

and three one-forms associated to a three-sphere:

$$\begin{aligned}
w^1 &= \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,,\\
w^2 &= -\cos\psi\sin\theta_2 d\phi_2 + \sin\psi d\theta_2\,,\\
w^3 &= d\psi + \cos\theta_2 d\phi_2\,.
\end{aligned} \tag{2.10}$$

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After an straightforward calculation one can verify that these forms satisfy

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\,\sigma^j \wedge \sigma^k\,, \qquad dw^i = \frac{1}{2}\epsilon_{ijk}\,w^j \wedge w^k\,. \tag{2.11}$$

Moreover, the $T^{1,1}$ metric (2.1) can be rewritten as

$$ds^2_{T^{1,1}} = \frac{1}{6}\,(\,(\sigma^1)^2 + (\sigma^2)^2 + (w^1)^2 + (w^2)^2\,) + \frac{1}{9}\,(\,w^3 + \sigma^3\,)^2\,. \tag{2.12}$$

This form of writing the $T^{1,1}$ metric is the one that arises naturally when the conifold geometry is obtained [21] in the framework of the eight-dimensional gauged supergravity obtained from a Scherk-Schwarz reduction of eleven dimensional supergravity on a SU(2) group manifold [22]. In this approach one starts with a domain wall problem in eight dimensions and looks for BPS solutions of the equations of motion. These solutions are subsequently uplifted to eleven dimensions, where they represent gravity duals of branes wrapping non-trivial cycles. The topological twist needed to realize supersymmetry with wrapped branes is implemented in this approach in a very natural way. As shown in ref. [21], the conifold metric is obtained as the gravity dual of D6-branes wrapping a holomorphic S^2 inside a K3 manifold. Moreover, from the consistency of the reduction, the Killing spinors should not depend on the coordinates of the group manifold and, actually, in the one-form basis we will use they do not depend on any angular coordinate of the $T^{1,1}$ space. Accordingly, let us consider the following frame for the ten-dimensional metric (2.5):

$$\begin{aligned}
e^{x^\alpha} &= \frac{r}{L}\,dx^\alpha\,, \qquad (\alpha = 0,1,2,3)\,, \qquad e^r = \frac{L}{r}\,dr\,,\\
e^i &= \frac{L}{\sqrt{6}}\,\sigma^i\,, \qquad (i = 1,2)\,,\\
e^{\hat{i}} &= \frac{L}{\sqrt{6}}\,w^i\,, \qquad (i = 1,2)\,,\\
e^{\hat{3}} &= \frac{L}{3}\,(\,w^3 + \sigma^3\,)\,.
\end{aligned} \tag{2.13}$$

Let us also define the matrix Γ_* as:

$$\Gamma_* \equiv i\Gamma_{x^0x^1x^2x^3}\,. \tag{2.14}$$

Then, the Killing spinors for the type IIB background (2.5) take the following form:

$$\epsilon = r^{\frac{\Gamma_*}{2}}\,\left(1 + \frac{\Gamma_r}{2L^2}\,x^\alpha\,\Gamma_{x^\alpha}\,(1 - \Gamma_*\,)\right)\,\eta\,, \tag{2.15}$$

where η is a constant spinor satisfying

$$\Gamma_{12}\,\eta = i\eta\,, \qquad \Gamma_{\hat{1}\hat{2}}\,\eta = -i\eta\,. \tag{2.16}$$

In eq. (2.15) we are parametrizing the dependence of ϵ on the AdS_5 coordinates as in ref. [26]. Notice that, as the matrix multiplying η in eq. (2.15) commutes with Γ_{12} and $\Gamma_{\hat{1}\hat{2}}$, the spinor ϵ also satisfies the conditions (2.16), namely:

$$\Gamma_{12}\,\epsilon = i\epsilon\,, \qquad \Gamma_{\hat{1}\hat{2}}\,\epsilon = -i\epsilon\,. \tag{2.17}$$

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It is clear from eqs. (2.15) and (2.16) that our system is 1/4 supersymmetric, i.e. it preserves 8 supersymmetries, as it corresponds to the supergravity dual of a $\mathcal{N}=1$ superconformal field theory in four dimensions. Moreover, let us decompose the constant spinor η according to the different eigenvalues of the matrix Γ_*:

$$\Gamma_*\,\eta_\pm = \pm\eta_\pm\,. \tag{2.18}$$

Using this decomposition in eq. (2.15) we obtain two types of Killing spinors

$$\begin{aligned} \epsilon_+ &= r^{1/2}\,\eta_+\,, \\ \epsilon_- &= r^{-1/2}\,\eta_- + \frac{r^{1/2}}{L^2}\,\Gamma_r\, x^\alpha \Gamma_{x^\alpha}\,\eta_-\,. \end{aligned} \tag{2.19}$$

Notice that the four spinors ϵ_+ are independent of the coordinates x^α and $\Gamma_*\,\epsilon_+ = \epsilon_+$. On the contrary, the ϵ_-'s do depend on the x^α's and are not eigenvectors of Γ_*. The latter correspond to the four superconformal supersymmetries, while the ϵ_+'s are the ones corresponding to the ordinary ones.

It is also interesting to write the form of the Killing spinors when global coordinates are used for the AdS_5 part of the metric. In these coordinates the ten-dimensional metric takes the form:

$$ds^2_{10} = L^2\,\left[-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right] + L^2\, ds^2_{T^{1,1}}\,, \tag{2.20}$$

where $d\Omega_3^2$ is the metric of a unit three-sphere parametrized by three angles $(\alpha^1, \alpha^2, \alpha^3)$:

$$d\Omega_3^2 = (d\alpha^1)^2 + \sin^2\alpha^1\,\left((d\alpha^2)^2 + \sin^2\alpha^2\,(d\alpha^3)^2\right)\,, \tag{2.21}$$

with $0 \le \alpha^1, \alpha^2 \le \pi$ and $0 \le \alpha^3 \le 2\pi$. In order to write down the Killing spinors in these coordinates, let us choose the following frame for the AdS_5 part of the metric:

$$\begin{aligned} e^t &= L\cosh\rho\, dt\,, \qquad e^\rho = L d\rho\,, \\ e^{\alpha^1} &= L\sinh\rho\, d\alpha^1\,, \\ e^{\alpha^2} &= L\sinh\rho\,\sin\alpha^1\, d\alpha^2\,, \\ e^{\alpha^3} &= L\sinh\rho\,\sin\alpha^1\,\sin\alpha^2\, d\alpha^3\,. \end{aligned} \tag{2.22}$$

We will continue to use the same frame forms as in eq. (2.13) for the $T^{1,1}$ part of the metric. If we now define the matrix

$$\gamma_* \equiv \Gamma_t\,\Gamma_\rho\,\Gamma_{\alpha^1\alpha^2\alpha^3}\,, \tag{2.23}$$

then, the Killing spinors in these coordinates can be written as [27]

$$\epsilon = e^{-i\frac{\rho}{2}\Gamma_\rho\gamma_*}\, e^{-i\frac{t}{2}\Gamma_t\gamma_*}\, e^{-\frac{\alpha^1}{2}\Gamma_{\alpha^1\rho}}\, e^{-\frac{\alpha^2}{2}\Gamma_{\alpha^2\alpha^1}}\, e^{-\frac{\alpha^3}{2}\Gamma_{\alpha^3\alpha^2}}\,\eta\,, \tag{2.24}$$

where η is a constant spinor which satisfies the same conditions as in eq. (2.16).

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2.2 Supersymmetric probes

Let us consider a Dp-brane probe in the KW background (2.5) and let ξ^μ ($\mu = 0, \cdots, p$) be a set of worldvolume coordinates. If X^M denote ten-dimensional coordinates, the Dp-brane embedding will be characterized by a set of functions $X^M(\xi^\mu)$. The induced metric on the worldvolume is

$$g_{\mu\nu} = \partial_\mu X^M \, \partial_\nu X^N \, G_{MN} \, , \tag{2.25}$$

where G_{MN} is the ten-dimensional metric. Let us denote by $E^{\underline{M}}_N$ the coefficients that appear in the expression of the frame one-forms $e^{\underline{M}}$ of the ten-dimensional metric in terms of the differentials of the coordinates, namely:

$$e^{\underline{M}} = E^{\underline{M}}_N \, dX^N \, . \tag{2.26}$$

Then, the induced Dirac matrices on the worldvolume are defined as

$$\gamma_\mu = \partial_\mu \, X^M \, E^{\underline{N}}_M \, \Gamma_{\underline{N}} \, , \tag{2.27}$$

where $\Gamma_{\underline{N}}$ are constant ten-dimensional Dirac matrices. Moreover, the pullback of the frame one-forms $e^{\underline{M}}$ is given by

$$P[\, e^{\underline{M}} \,] = E^{\underline{M}}_N \partial_\mu \, X^N \, d\xi^\mu \, \equiv C^{\underline{M}}_\mu \, d\xi^\mu \, , \tag{2.28}$$

where, in the last step, we have defined the coefficients $C^{\underline{M}}_\mu \equiv E^{\underline{M}}_N \partial_\mu \, X^N$. Notice that the induced Dirac matrices γ_μ can be expressed in terms of the constant Γ's by means of these same coefficients $C^{\underline{M}}_\mu$, namely:

$$\gamma_\mu = C^{\underline{M}}_\mu \, \Gamma_{\underline{M}} \, . \tag{2.29}$$

Let us now decompose the complex spinor ϵ used up to now in its real and imaginary parts as $\epsilon = \epsilon_1 + i\epsilon_2$. We can now arrange the two Majorana-Weyl spinors ϵ_1 and ϵ_2 as a two-dimensional vector $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$. Acting on these real two-component spinors, the kappa symmetry matrix of a Dp-brane in the type IIB theory is given by [18]:

$$\Gamma_\kappa = \frac{1}{(p+1)!\sqrt{-g}} \, \epsilon^{\mu_1 \cdots \mu_{p+1}} \, (\tau_3)^{\frac{p-3}{2}} \, i\tau_2 \, \otimes \, \gamma_{\mu_1 \cdots \mu_{p+1}} \, , \tag{2.30}$$

where g is the determinant of the induced metric $g_{\mu\nu}$, the τ_i ($i = 1, 2, 3$) are Pauli matrices that act on the two-dimensional vector $\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ and $\gamma_{\mu_1 \cdots \mu_{p+1}}$ denotes the antisymmetrized product of the induced gamma matrices (2.29). In eq. (2.30) we have assumed that there are not worldvolume gauge fields on the Dp-brane and we have taken into account that the Neveu-Schwarz B field is zero for the Klebanov-Witten background. The absence of worldvolume gauge fields is consistent with the equations of motion of the probe if there are not source terms for the worldvolume gauge field in the action. These source terms must be linear in the gauge field and they can only come from the Wess-Zumino part of the Dirac-Born-Infeld lagrangian. The former is responsible for the coupling of the probe to the Ramond-Ramond fields of the background. In our case we have only one of such Ramond-Ramond fields, namely the selfdual five-form $F^{(5)}$. If we denote by $C^{(4)}$ its potential ($F^{(5)} = dC^{(4)}$), it is clear that the only term linear in the worldvolume gauge

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field A in the Wess-Zumino lagrangian is:

$$\int F\wedge C^{(4)} = \int A\wedge F^{(5)}\,, \tag{2.31}$$

where $F = dA$ and we have integrated by parts. In eq. (2.31) it is understood that the pullback of the Ramond-Ramond fields to the worldvolume is being taken. By counting the degree of the form under the integral in eq. (2.31), it is obvious that such a term can only exist for a D5-brane and it is zero if the brane worldvolume does not capture the flux of the $F^{(5)}$. As can be easily checked by inspection, this happens in all the cases studied in this paper except for the baryon vertex configuration analyzed in appendix A. In this case, the expression (2.30) for the kappa symmetry matrix is not valid and one has to use the more general formula given in ref. [18].

Nevertheless, we could try to find embeddings with non-vanishing worldvolume gauge fields even when the equations of motion allow to put them to zero. For simplicity, in this paper we would not try to do this, except for the case studied in subsection A.3 of appendix A, where a supersymmetric embedding of a D5-brane with flux of the worldvolume gauge field is obtained.

The supersymmetric BPS configurations of the brane probe are obtained by requiring the condition:

$$\Gamma_\kappa\,\epsilon = \epsilon\,, \tag{2.32}$$

where ϵ is a Killing spinor of the background [19]. It follows from eq. (2.30) that Γ_κ depends on the induced metric and Dirac matrices, which in turn are determined by the D-brane embedding $X^M(\xi^\mu)$. Actually, eq. (2.32) should be regarded as an equation whose unknowns are both the embedding $X^M(\xi^\mu)$ and the Killing spinors ϵ. The number of solutions for ϵ determines the amount of background supersymmetry that is preserved by the probe. Notice that we have written Γ_κ in eq. (2.30) as a matrix acting on real two-component spinors, while we have written the Killing spinors of the background in complex notation. However, it is straightforward to find the following rules to pass from complex to real spinors:

$$\epsilon^*\,\leftrightarrow\,\tau_3\,\epsilon\,,\qquad i\epsilon^*\,\leftrightarrow\,\tau_1\,\epsilon\,,\qquad i\epsilon\,\leftrightarrow\,-i\tau_2\,\epsilon\,. \tag{2.33}$$

As an example of the application of these rules, notice that the projections (2.17), satisfied by the Killing spinors of the $AdS_5\times T^{1,1}$ background, can be written as:

$$\Gamma_{12}\,\otimes\,i\tau_2\,\epsilon = -\Gamma_{\hat{1}\hat{2}}\,\otimes\,i\tau_2\,\epsilon = \epsilon\,. \tag{2.34}$$

Let us now discuss the general strategy to solve the kappa symmetry equation (2.32). First of all, notice that, by using the explicit form (2.29) of the induced Dirac matrices in the expression of Γ_κ (eq. (2.30)), eq. (2.32) takes the form:

$$\sum_i c_i\,\Gamma^{(i)}_{AdS_5}\,\Gamma^{(i)}_{T^{1,1}}\,\otimes\,(\tau_3)^{\frac{p-3}{2}}\,i\tau_2\,\epsilon = \epsilon\,, \tag{2.35}$$

where $\Gamma^{(i)}_{AdS_5}$ ($\Gamma^{(i)}_{T^{1,1}}$) are antisymmetrized products of constant ten-dimensional Dirac matrices along the AdS_5 ($T^{1,1}$) directions and the coefficients c_i depend on the embedding $X^M(\xi^\mu)$ of the Dp-brane in the $AdS_5\times T^{1,1}$ space. Actually, due to the relations (2.34)

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satisfied by the Killing spinors ϵ, some of the terms in eq. (2.35) are not independent. After expressing eq. (2.35) as a sum of independent contributions, we obtain a new projection for the Killing spinor ϵ. This projection is not, in general, consistent with the conditions (2.34) since some of the matrices appearing on the left-hand side of eq. (2.35) do not commute with those appearing in (2.34). The only way of making eqs. (2.34) and (2.35) consistent with each other is by requiring the vanishing of the coefficients c_i of these non-commuting matrices, which gives rise to a set of first-order BPS differential equations for the embedding $X^M(\xi^\mu)$.

Notice that the kappa symmetry projection of the BPS configurations must be satisfied at any point of the worldvolume of the brane probe. However, the Killing spinors ϵ do depend on the coordinates (see eqs. (2.15) or (2.24)). Thus, it is not obvious at all that the $\Gamma_\kappa \epsilon = \epsilon$ condition can be imposed at all points of the worldvolume. This fact would be guaranteed if we could recast eq. (2.32) for BPS configurations as an algebraic condition on the constant spinor η of eqs. (2.15) or (2.24). This algebraic condition on η must involve a constant matrix projector and its fulfillment is generically achieved by imposing some extra conditions to the spinor ϵ (which reduces the amount of supersymmetry preserved by the configuration) or by restricting appropriately the embedding. For example, when working on the coordinates (2.6), one should check whether the kappa symmetry projector commutes with the matrix Γ_* of eq. (2.14). If this is the case, one can consider spinors such as the ϵ_+'s of eq. (2.19), which are eigenvectors of Γ_* and, apart from an irrelevant factor depending on the radial coordinate, are constant. In case we use the parametrization (2.24), we should check that, for the BPS embeddings, the kappa symmetry projection commutes with the matrix multiplying the spinor η on the right-hand side of eq. (2.24).

If the BPS differential equations can be solved, one should verify that the corresponding configuration also solves the equations of motion derived from the Dirac-Born-Infeld action of the probe. In all the cases analyzed in this paper the solutions of the BPS equations also solve the equations of motion. Actually, we will verify that these BPS configurations saturate a bound for the energy, as is expected for a supersymmetric worldvolume soliton.

3. Kappa symmetry for a D3-brane probe

As our first example of D-brane probe in the Klebanov-Witten background, let us consider a D3-brane. By particularizing eq. (2.30) to this $p = 3$ case, we obtain that Γ_κ is given by:

$$\Gamma_\kappa = -\frac{i}{4!\sqrt{-g}}\,\epsilon^{\mu_1\cdots\mu_4}\,\gamma_{\mu_1\cdots\mu_4}\,, \tag{3.1}$$

where we have used the dictionary (2.33) to obtain the expression of Γ_κ acting on complex spinors.

We will consider several possible configurations with different number of dimensions on which the D3-brane is wrapped. Since the $T^{1,1}$ space is topologically $S^2 \times S^3$, it is natural to consider branes wrapped over three- and two- cycles. We will study first the case of D3-branes wrapped over a three-dimensional manifold, where we will find a rich set of BPS configurations. In appendix A we will allow the D3-brane to be extended along

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one spacelike direction of the AdS_5 and wrapped over a two-cycle of the $T^{1,1}$ coset. In this case we will not be able to find BPS embeddings of the D3-brane probe. However, we will verify in appendix A that there exist stable, non-supersymmetric, embeddings of D3-branes wrapping a two-cycle. Actually (see section A.1), these two-cycles are just the ones obtained in section 4.1, i.e. those over which a D5-brane can be wrapped supersymmetrically.

3.1 D3-branes wrapped on a three-cycle

Let us use global coordinates as in eq. (2.20) for the AdS_5 part of the metric. We will search for supersymmetric configurations which are pointlike from the AdS_5 point of view and wrap a compact three-manifold within $T^{1,1}$. Accordingly, let us take the following set of worldvolume coordinates:

$$\xi^{\mu} = (\, t, \theta_1, \phi_1, \psi \,)\,, \tag{3.2}$$

and consider embeddings of the type:

$$\theta_2 = \theta_2(\theta_1, \phi_1)\,, \qquad \phi_2 = \phi_2(\theta_1, \phi_1)\,, \tag{3.3}$$

with the radial coordinate ρ and the angles α^i being constant. For these embeddings Γ_κ in eq. (3.1) reduces to:

$$\Gamma_\kappa = -iL\,\frac{\cosh\rho}{\sqrt{-g}}\,\Gamma_t\,\gamma_{\theta_1\phi_1\psi}\,. \tag{3.4}$$

The induced gamma matrices along the worldvolume coordinates can be readily obtained from the general expression (2.29). The result is:

$$\begin{aligned}
\gamma_{\theta_1} &= \frac{L}{\sqrt{6}}\Big[\Gamma_1 + (\cos\psi\,\partial_{\theta_1}\theta_2 + \sin\psi\sin\theta_2\partial_{\theta_1}\phi_2\,)\Gamma_{\hat 1} + \\
&\qquad + (\sin\psi\,\partial_{\theta_1}\theta_2 - \cos\psi\sin\theta_2\partial_{\theta_1}\phi_2)\Gamma_{\hat 2}\Big] + \frac{L}{3}\cos\theta_2\,\partial_{\theta_1}\phi_2\Gamma_{\hat 3}\,,\\
\gamma_{\phi_1} &= \frac{L}{\sqrt{6}}\Big[\sin\theta_1\,\Gamma_2 + (\,\sin\theta_2\,\sin\psi\,\partial_{\phi_1}\phi_2 + \cos\psi\,\partial_{\phi_1}\theta_2\,)\Gamma_{\hat 1} + \\
&\qquad + (\sin\psi\,\partial_{\phi_1}\theta_2 - \cos\psi\,\sin\theta_2\,\partial_{\phi_1}\phi_2)\Gamma_{\hat 2}\Big] + \frac{L}{3}(\cos\theta_1 + \cos\theta_2\partial_{\phi_1}\phi_2)\Gamma_{\hat 3}\,,\\
\gamma_\psi &= \frac{L}{3}\,\Gamma_{\hat 3}\,.
\end{aligned} \tag{3.5}$$

By using these expressions and the projections (2.17), it is easy to verify that:

$$\frac{18}{L^3}\,\gamma_{\theta_1\phi_1\psi}\,\epsilon = ic_1\Gamma_{\hat 3}\epsilon + (c_2 + ic_3)\,e^{-i\psi}\,\Gamma_{1\hat 2\hat 3}\,\epsilon\,, \tag{3.6}$$

with the coefficients c_1, c_2 and c_3 being:

$$\begin{aligned}
c_1 &= \sin\theta_1 + \sin\theta_2\,(\partial_{\theta_1}\theta_2\,\partial_{\phi_1}\phi_2 - \partial_{\theta_1}\phi_2\,\partial_{\phi_1}\theta_2\,)\,,\\
c_2 &= \sin\theta_1\,\partial_{\theta_1}\theta_2 - \sin\theta_2\partial_{\phi_1}\phi_2\,,\\
c_3 &= \partial_{\phi_1}\theta_2 + \sin\theta_1\,\sin\theta_2\,\partial_{\theta_1}\phi_2\,.
\end{aligned} \tag{3.7}$$

Following the general strategy discussed at the end of section 2.2, we have to ensure that the kappa symmetry projection $\Gamma_\kappa\epsilon = \epsilon$ is compatible with the conditions (2.17). By

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inspecting the right-hand side of eq. (3.6) it is fairly obvious that the terms containing the matrix $\Gamma_{1\hat{2}\hat{3}}$ would give rise to contributions not compatible with the projection (2.17). Thus, it is clear that to have $\Gamma_\kappa\,\epsilon=\epsilon$ we must impose the condition

$$c_2=c_3=0\,, \tag{3.8}$$

which yields the following differential equation for $\theta_2(\theta_1,\phi_1)$ and $\phi_2(\theta_1,\phi_1)$:

$$\begin{aligned}\sin\theta_1\,\partial_{\theta_1}\theta_2 &= \sin\theta_2\,\partial_{\phi_1}\phi_2\,,\\ \partial_{\phi_1}\theta_2 &= -\sin\theta_1\sin\theta_2\,\partial_{\theta_1}\phi_2\,.\end{aligned} \tag{3.9}$$

We will prove below that the first-order equations (3.9), together with some extra condition on the Killing spinor ϵ, are enough to ensure that $\Gamma_\kappa\epsilon=\epsilon$, i.e. that our D3-brane probe configuration preserves some fraction of supersymmetry. For this reason we will refer to (3.9) as the BPS equations of the embedding. It is clear from eq. (3.4) that, in order to compute Γ_κ, we need to calculate the determinant g of the induced metric. From eq. (2.25) and the explicit form (3.3) of our ansatz, it is easy to verify that the non-vanishing elements of the induced metric are:

$$\begin{aligned}g_{\tau\tau} &= -L^2\cosh^2\rho\,,\\ g_{\theta_1\theta_1} &= \frac{L^2}{6}\Big[1+(\partial_{\theta_1}\theta_2)^2+\sin^2\theta_2(\partial_{\theta_1}\phi_2)^2\Big]+\frac{L^2}{9}\cos^2\theta_2(\partial_{\theta_1}\phi_2)^2\,,\\ g_{\phi_1\phi_1} &= \frac{L^2}{6}\Big[\sin^2\theta_1+(\partial_{\phi_1}\theta_2)^2+\sin^2\theta_2(\partial_{\phi_1}\phi_2)^2\Big]+\frac{L^2}{9}(\cos\theta_1+\cos\theta_2\partial_{\phi_1}\phi_2)^2\,,\\ g_{\psi\psi} &= \frac{L^2}{9}\,,\\ g_{\phi_1\theta_1} &= \frac{L^2}{6}\Big[\partial_{\theta_1}\theta_2\partial_{\phi_1}\theta_2+\sin^2\theta_2\partial_{\theta_1}\phi_2\partial_{\phi_1}\phi_2\Big]+\frac{L^2}{9}\cos\theta_2\partial_{\theta_1}\phi_2(\cos\theta_1+\cos\theta_2\partial_{\phi_1}\phi_2)\,,\\ g_{\phi_1\psi} &= \frac{L^2}{9}(\cos\theta_1+\cos\theta_2\partial_{\phi_1}\phi_2)\,,\\ g_{\theta_1\psi} &= \frac{L^2}{9}\cos\theta_2\partial_{\theta_1}\phi_2\,.\end{aligned} \tag{3.10}$$

Let us now define

$$\begin{aligned}\alpha &\equiv \frac{L^2}{6}\Big[1+(\partial_{\theta_1}\theta_2)^2+\sin^2\theta_2\,(\partial_{\theta_1}\phi_2)^2\Big]\,,\\ \beta &\equiv \frac{L^2}{6}\Big[\sin^2\theta_1+(\partial_{\phi_1}\theta_2)^2+\sin^2\theta_2\,(\partial_{\phi_1}\phi_2)^2\Big]\,,\\ \gamma &\equiv \frac{L^2}{6}\Big[\partial_{\theta_1}\theta_2\,\partial_{\phi_1}\theta_2+\sin^2\theta_2\,\partial_{\theta_1}\phi_2\,\partial_{\phi_1}\phi_2\Big]\,.\end{aligned} \tag{3.11}$$

From these values one can prove that

$$\sqrt{-g}=\frac{L^2\cosh\rho}{3}\,\sqrt{\alpha\beta-\gamma^2}\,. \tag{3.12}$$

Moreover, if the BPS equations (3.9) are satisfied, the functions α, β and γ take the values:

$$\alpha_{|_{BPS}}=\frac{L^2}{6\sin\theta_1}\,c_{1|_{BPS}}\,,\qquad \beta_{|_{BPS}}=\frac{L^2\sin\theta_1}{6}\,c_{1|_{BPS}}\,,\qquad \gamma_{|_{BPS}}=0\,, \tag{3.13}$$

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where c_1 is written in eq. (3.7) and the determinant of the induced metric is

$$\sqrt{-g}_{|_{BPS}} = \frac{L^4}{18} \cosh\rho\, c_{1|_{BPS}} \,. \tag{3.14}$$

From this expression of $\sqrt{-g}_{|_{BPS}}$ it is straightforward to verify that, if the first-order system (3.9) holds, one has:

$$\Gamma_\kappa\, \epsilon = \Gamma_t \Gamma_{\hat{3}}\, \epsilon \,. \tag{3.15}$$

Thus, the condition $\Gamma_\kappa\, \epsilon = \epsilon$ is equivalent to

$$\Gamma_t \Gamma_{\hat{3}}\, \epsilon = \epsilon \,. \tag{3.16}$$

Let us now plug in this equation the explicit form (2.24) of the Killing spinors. Notice that, except for $\Gamma_\rho\, \gamma_*$, $\Gamma_t\Gamma_{\hat{3}}$ commutes with all matrices appearing on the right-hand side of eq. (2.24). Actually, only for $\rho = 0$ the coefficient of $\Gamma_\rho\, \gamma_*$ in (2.24) vanishes and, thus, only at this point of AdS_5 the equation $\Gamma_\kappa\, \epsilon = \epsilon$ can be satisfied. In this case, it reduces to the following condition on the constant spinor η:

$$\Gamma_t \Gamma_{\hat{3}}\, \eta = \eta \,. \tag{3.17}$$

Then, in order to have a supersymmetric embedding, we must place our D3-brane probe at $\rho = 0$, i.e. at the center of the AdS_5 space. The resulting configuration is 1/8 supersymmetric: it preserves four Killing spinors of the type (2.24) with $\Gamma_{12}\, \eta = -\Gamma_{\hat{1}\hat{2}}\, \eta = i\eta$, $\Gamma_t \Gamma_{\hat{3}}\, \eta = \eta$.

3.1.1 Integration of the first-order equations

Let us now integrate the first-order differential equations (3.9). Remarkably, this same set of equations has been obtained in ref. [28] in the study of the supersymmetric embeddings of D5-brane probes in the Maldacena-Núñez background [29]. It was shown in ref. [28] that, after a change of variables, the pair of eqs. in (3.9) can be converted into the Cauchy-Riemann equations. Indeed, let us define two new variables u_1 and u_2, related to θ_1 and θ_2 as follows:

$$u_1 = \log\left(\tan\frac{\theta_1}{2}\right), \qquad u_2 = \log\left(\tan\frac{\theta_2}{2}\right). \tag{3.18}$$

Then, it is straightforward to demonstrate that the equations (3.9) can be written as:

$$\frac{\partial u_2}{\partial u_1} = \frac{\partial \phi_2}{\partial \phi_1}\,, \qquad \frac{\partial u_2}{\partial \phi_1} = -\frac{\partial \phi_2}{\partial u_1}\,, \tag{3.19}$$

i.e. as the Cauchy-Riemann equations for the variables (u_1, ϕ_1) and (u_2, ϕ_2). Since $u_1, u_2 \in (-\infty, +\infty)$ and $\phi_1, \phi_2 \in (0, 2\pi)$, the above equations are actually the Cauchy-Riemann equations in a band. The general integral of these equations is obtained by requiring that $u_2 + i\phi_2$ be an arbitrary function of the holomorphic variable $u_1 + i\phi_1$:

$$u_2 + i\phi_2 = f(u_1 + i\phi_1)\,. \tag{3.20}$$

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Let us now consider the particular case in which $u_2 + i\phi_2$ depends linearly on $u_1 + i\phi_1$, namely:

$$u_2 + i\phi_2 = m(u_1 + i\phi_1) + \text{constant}\,, \tag{3.21}$$

where m is constant. Let us further assume that m is real and integer. By equating the imaginary parts of both sides of eq. (3.21), one gets:

$$\phi_2 = m\,\phi_1 + \text{constant}\,. \tag{3.22}$$

Clearly, m can be interpreted as a winding number [28]. Moreover, from the real part of eq. (3.21) we immediately obtain u_2 as a function of u_1 for this embedding. By using the change of variables of eq. (3.18) we can convert this $u_2 = u_2(u_1)$ function in a relation between the angles θ_1 and θ_2, namely:

$$\tan\frac{\theta_2}{2} = C\,\left(\tan\frac{\theta_1}{2}\right)^m\,, \tag{3.23}$$

with C constant. Following ref. [28] we will call m-winding embedding to the brane configuration corresponding to eqs. (3.22) and (3.23). Notice that for $m = 0$ the above solution reduces to $\theta_2 = $ constant, $\phi_2 = $ constant. This zero-winding configuration of the D3-brane is just the one proposed in ref. [7] as dual to the dibaryon operators of the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge theory. Moreover, when $m = \pm 1$ we have the so-called unit-winding embeddings. When the constant C in eq. (3.23) is equal to one, it is easy to find the following form of these unit-winding configurations:

$$\begin{aligned} &\theta_2 = \theta_1\,, \qquad \phi_2 = \phi_1\,, \qquad (m = 1)\,,\\ &\theta_2 = \pi - \theta_1\,, \qquad \phi_2 = 2\pi - \phi_1\,, \qquad (m = -1)\,, \end{aligned} \tag{3.24}$$

where we have adjusted appropriately the constant of eq. (3.22). Notice that the two possibilities in (3.24) correspond to the two possible identifications of the two (θ_1, ϕ_1) and (θ_2, ϕ_2) two-spheres.

3.1.2 Holomorphic structure

It is also interesting to write the m-winding embeddings just found in terms of the holomorphic coordinates $z_1, \cdots, z_4$ of the conifold. Actually, by inspecting eq. (2.4), and comparing it with the functions $\theta_2 = \theta_2(\theta_1)$ and $\phi_2 = \phi_2(\phi_1)$ corresponding to a m-winding embedding (eqs. (3.22) and (3.23)), one concludes that the latter can be written, for example, as:[1]

$$\frac{z_1}{z_4} = C\,\left(\frac{z_1}{z_3}\right)^m\,. \tag{3.25}$$

Thus, the m-winding embeddings of the D3-brane in the $T^{1,1}$ space can be characterized as the vanishing locus of a polynomial in the z_i coordinates of $\mathbb{C}^4$. In order to find this polynomial in its full generality, let us consider the solutions of the following polynomial

[1] If the function f in eq. (3.20) satisfies that $\bar{f}(z) = f(\bar{z})$, then the general solution (3.20) can be written as $\log\frac{z_1}{z_4} = f\Big[\log\frac{z_1}{z_3}\Big]$.

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equation

$$z_1^{m_1}\, z_2^{m_2}\, z_3^{m_3}\, z_4^{m_4} = \text{constant}\,, \tag{3.26}$$

where the $m_i's$ are real constants and we will assume that:

$$\begin{aligned} m_1 + m_2 + m_3 + m_4 &= 0\,, \\ m_1 + m_3 &\neq 0\,. \end{aligned} \tag{3.27}$$

By plugging the representation (2.3) of the z_i coordinates in the left-hand side of eq. (3.26), one readily proves that, due to the first condition in eq. (3.27), r and ψ are not restricted by eq. (3.26). Moreover, by looking at the phase of the left-hand side of eq. (3.26) one realizes that, if the second condition in (3.27) holds, ϕ_2 is related to ϕ_1 as in eq. (3.22), with m being given by:

$$m = \frac{m_2 + m_3}{m_1 + m_3} = \frac{m_1 + m_4}{m_2 + m_4}\,. \tag{3.28}$$

Furthermore, from the modulus of eq. (3.26) we easily prove that $\theta_2(\theta_1)$ is indeed given by eq. (3.23) with the winding m displayed in eq. (3.28). As a check of these identifications let us notice that, by using the conifold equation (2.2), the embedding (3.26) is invariant under the change

$$(m_1, m_2, m_3, m_4) \rightarrow (m_1 - n, m_2 - n, m_3 + n, m_4 + n) \tag{3.29}$$

for arbitrary n. Our relation (3.28) of the winding m and the exponents m_i is also invariant under the change (3.29).

As particular cases notice that the zero-winding embedding can be described by the equation $z_1 = C z_4$, while the unit-winding solution corresponds to $z_3 = C\, z_4$ for $m = 1$ and $z_1 = C\, z_2$ for $m = -1$.

For illustrative purposes, let us consider the holomorphic structure of the solutions (3.20) with a non-linear function f. It is easy to see that, if the function f is not linear, the corresponding holomorphic equation is non-polynomial. For example, the embedding $u_2 + i\phi_2 = (u_1 + i\phi_1)^2$ corresponds to the equation $\frac{z_1}{z_4} = \exp[\log^2 \frac{z_1}{z_3}]$. Contrary to what happens to the solutions (3.26) (see subsection 3.1.4), the field theory dual of these non-polynomial embeddings is completely unclear for us and we will not pursue their study here.

3.1.3 Energy bound

The Dirac-Born-Infeld lagrangian density for the D3-brane probe is given by

$$\mathcal{L} = -\sqrt{-g}\,, \tag{3.30}$$

where we have taken the D3-brane tension equal to one and the value of $\sqrt{-g}$ for a general embedding of the type (3.3) has been written in eq. (3.12). We have checked by explicit calculation that any solution of the first-order equations (3.9) also satisfies the Euler-Lagrange equations of motion derived from the lagrangian (3.30). Moreover, the hamiltonian density for the static configurations we are considering is just $\mathcal{H} = -\mathcal{L}$. We

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are going to prove that $\mathcal{H}$ satisfies a bound which is saturated just when the embedding satisfies the BPS equations (3.9). To check this fact, let us consider arbitrary functions $\theta_2(\theta_1, \phi_1)$ and $\phi_2(\theta_1, \phi_1)$. For an embedding at $\rho = 0$, we can write:

$$\mathcal{H} = \frac{L^2}{3} \sqrt{\alpha\beta - \gamma^2}\,, \tag{3.31}$$

where α, β and γ are given in eq. (3.11) and we have used the value of $\sqrt{-g}$ given in eq. (3.12). Let us now rewrite $\mathcal{H}$ as $\mathcal{H} = |\mathcal{Z}| + \mathcal{S}$, where $\mathcal{Z} = \frac{L^4}{18}\, c_1$, with c_1 given in the first expression in eq. (3.7). It is easily checked that $\mathcal{Z}$ can be written as a total derivative

$$\mathcal{Z} = \partial_{\theta_1}\, \mathcal{Z}^{\theta_1} + \partial_{\phi_1}\, \mathcal{Z}^{\phi_1}\,, \tag{3.32}$$

with

$$\mathcal{Z}^{\theta_1} = -\frac{L^4}{18}\,(\cos\theta_1 + \cos\theta_2\, \partial_{\phi_1}\phi_2)\,, \qquad \mathcal{Z}^{\phi_1} = \frac{L^4}{18}\, \cos\theta_2\, \partial_{\theta_1}\phi_2\,. \tag{3.33}$$

Clearly, $\mathcal{S}$ is given by:

$$\mathcal{S} = \frac{L^2}{3} \sqrt{\alpha\beta - \gamma^2} - \frac{L^4}{18}\, |c_1|\,. \tag{3.34}$$

Let us now prove that $\mathcal{S} \geq 0$ for an arbitrary embedding of the type (3.3). Notice that this is equivalent to the following bound

$$\mathcal{H} \geq |\mathcal{Z}|\,. \tag{3.35}$$

Moreover , it is easy to verify that the condition $\mathcal{S} \geq 0$ is equivalent to:

$$(\sin\theta_1\, \partial_{\theta_1}\theta_2 - \sin\theta_2 \partial_{\phi_1}\phi_2)^2 + (\partial_{\phi_1}\theta_2 + \sin\theta_1 \sin\theta_2\, \partial_{\theta_1}\phi_2)^2 \geq 0\,, \tag{3.36}$$

which is obviously satisfied for arbitrary functions $\theta_2(\theta_1, \phi_1)$ and $\phi_2(\theta_1, \phi_1)$. Moreover, by inspecting eq. (3.36) one easily concludes that the equality in (3.36) is equivalent to the first-order BPS equations (3.9) and, thus $\mathcal{H} = |\mathcal{Z}|$ for the BPS embeddings (actually, $\mathcal{Z} \geq 0$ if the BPS equations are satisfied).

3.1.4 Field theory dual

In this subsection we will give some hints on the field theory dual of a general D3-brane m-winding embedding. First of all, let us try to find the conformal dimension Δ of the corresponding operator in the dual field theory. According to the general AdS/CFT arguments, and taking into account the zero-mode corrections as in ref. [8], one should have

$$\Delta = LM\,, \tag{3.37}$$

where L is given in (2.5) and M is the mass of the wrapped D3-brane. The latter can be written simply as

$$M = T_3\, V_3\,, \tag{3.38}$$

where V_3 is the volume of the cycle computed with the induced metric (3.10) and T_3 is the tension of the D3-brane, given by

$$T_3 = \frac{1}{8\pi^3 (\alpha')^2 g_s}\,. \tag{3.39}$$

Taking into account the results of section 3.1.3, it is not difficult to compute the value of V_3 for the three-cycle $\mathcal{C}^{(m)}$ corresponding to the m-winding embedding. The result is:

$$V_3 = \frac{8\pi^2 L^3}{9}\,(1+|m|)\,. \tag{3.40}$$

Now, the mass of the wrapped D3-brane can be readily obtained, namely:

$$M = \frac{L^3}{9\pi(\alpha')^2 g_s}\,(1+|m|)\,. \tag{3.41}$$

By plugging this result in eq. (3.37), and using the value of L given in eq. (2.5), one obtains the following value of the conformal dimension Δ

$$\Delta = \frac{3}{4}\,(1+|m|)\,N\,. \tag{3.42}$$

Notice that for $m = 0$ we recover from (3.42) the result $\Delta = \frac{3}{4}N$ of ref. [7], which is the conformal dimension of an operator of the form A^N, with A being the $\mathcal{N} = 1$ chiral multiplets introduced in section 2 and a double antisymmetrization over the gauge indices of A is performed in the N^{th} power of A (for details see ref. [7]). Remember that the conformal dimensions of the A and B fields are $\Delta(A) = \Delta(B) = 3/4$. Thus, in view of the the result (3.42), it is natural to think that our wrapped D3-branes for general m correspond to operators with a field content of the form

$$(A^a B^b)^N\,, \qquad a+b = 1+|m|\,. \tag{3.43}$$

To determine the values of a and b in (3.43) one has to find the baryon number of the operator. Recall [6] that the U(1) baryon number symmetry acts on the $\mathcal{N} = 1$ matter multiplets as:

$$A_i \to e^{i\alpha}\,A_i\,, \qquad B_i \to e^{-i\alpha}\,B_i\,, \tag{3.44}$$

and, thus, the A (B) field has baryon number $+1(-1)$. Notice that the z_i coordinates are invariant under the transformation (3.44) of the homogeneous coordinates A_i and B_i (see eq. (2.7)). On the gravity side of the AdS/CFT correspondence, the baryon number (in units of N) can be identified with the third homology class of the three-cycle $\mathcal{C}^{(m)}$ over which the D3-brane is wrapped. Indeed, the third homology group of the $T^{1,1}$ space is $H_3(T^{1,1}) = \mathbb{Z}$. Moreover, the homology class of the cycle can be determined by representing it as the zero-locus of a polynomial in the A and B coordinates which transforms homogeneously under the U(1) symmetry (3.44). Actually, the charge of the polynomial under the baryon-number transformations (3.44) is just the class of the three-cycle $\mathcal{C}^{(m)}$ in $H_3(T^{1,1}) = \mathbb{Z}$.

It is easy to rewrite the results of section 3.1.2 in terms of the homogeneous coordinates A and B. Indeed, one can prove that the three-cycle $\mathcal{C}^{(m)}$ corresponding to the m-winding embedding can be written as

$$A_1\,B_2^m = c\,A_2\,B_1^m\,. \tag{3.45}$$

Notice that changing $m \to -m$ in eq. (3.45) is equivalent to the exchange $B_1 \leftrightarrow B_2$. Therefore, we can always arrange eq. (3.45) in such a way that the exponents of A and B

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are positive.[2] Moreover, the polynomial representing $\mathcal{C}^{(m)}$ transforms homogeneously under the symmetry (3.44) with charge $1-|m|$, which is just the class of $\mathcal{C}^{(m)}$ in $H_3(T^{1,1})=\mathbb{Z}$. One can confirm this result by computing the integral over $\mathcal{C}^{(m)}$ of the (pullback) of the three-form ω_3

$$\omega_3=\frac{1}{16\pi^2}\,d\psi\,(\,\sin\theta_1 d\theta_1 d\phi_1-\sin\theta_2 d\theta_2 d\phi_2)\,, \tag{3.46}$$

which has been suitably normalized. One can easily check that

$$\int_{\mathcal{C}^{(m)}}\omega_3=1-|m|\,, \tag{3.47}$$

which is the same result as that obtained by representing $\mathcal{C}^{(m)}$ as in eq. (3.45). Thus, the baryon number of the dual operator must be $(1-|m|)N$ and we have a new equation for the exponents a and b in (3.43), namely $a-b=1-|m|$, which allows to determine the actual values of a and b, i.e. $a=1$ and $b=|m|$. Thus, we are led to the conclusion that the field theory operator dual to our m-winding embedding must be of the form:

$$\Big(AB^{|m|}\Big)^N\,. \tag{3.48}$$

We will not attempt to determine here the gauge-invariant index structure of the operator with the field content (3.48) which is dual to the m-winding configurations of the D3-branes. Notice that, for generic values of m, the absolute value of the baryon number is greater than N, whereas for $m=\pm 1$ it vanishes. This last case resembles that of a giant graviton, although it is interesting to remember that our unit-winding embeddings are static, i.e. time-independent.

4. Kappa symmetry for a D5-brane probe

In this section we will explore the possibility of having supersymmetric configurations of D5-branes which wrap some cycle of the $T^{1,1}$ space. Notice that, according to the general expression of Γ_κ (eq. (2.30)) and to the dictionary of eq. (2.33), one has in this case:

$$\Gamma_\kappa\,\epsilon=\frac{i}{6!\,\sqrt{-g}}\,\epsilon^{\mu_1\cdots\mu_6}\,\gamma_{\mu_1\cdots\mu_6}\,\epsilon^*\,. \tag{4.1}$$

The complex conjugation of the right-hand side of eq. (4.1) will be of great importance in what follows. Recall that we want the D5-brane kappa symmetry projector to be compatible with the conditions $\Gamma_{12}\,\epsilon=i\epsilon$ and $\Gamma_{\hat{1}\hat{2}}\,\epsilon=-i\epsilon$ of eq. (2.17). Since in this D5-brane case the action of Γ_κ on ϵ involves the complex conjugation, this compatibility with the conditions (2.17) will force us to select embeddings for which the kappa symmetry projector mixes the two S^2 spheres, which will allow us to find the differential equations to be satisfied by the embedding.

[2] There is an obvious asymmetry in our equations between the A and B coordinates. The origin of this asymmetry is the particular choice of worldvolume coordinates we have made in (3.2). If we choose instead $\xi^\mu=(\,t,\theta_2,\phi_2,\psi\,)$ the role of A and B is exchanged. Alternatively, the same effect is obtained with the coordinates (3.2) by changing $m\to 1/m$.

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Actually, we will only be able to carry out successfully this program for the case of a D5-brane wrapped on a two-cycle. It has been proposed in ref. [7] that this kind of configurations represent a domain wall in the gauge theory side. Indeed, one of such D5-branes is an object of codimension one in AdS_5 and, as argued in ref. [7], upon crossing it the gauge theory group changes from $\mathrm{SU}(N)\times\mathrm{SU}(N)$ to $\mathrm{SU}(N)\times\mathrm{SU}(N+1)$. In the next subsection we will explain in detail how to find such supersymmetric embeddings and we will analyze some of their properties. In appendix A we will obtain another configuration preserving the same supersymmetry as the one considered in this section and we will study the effect of adding flux of the worldvolume gauge fields. In this appendix we also include a brief account of our unsuccessful attempts to find supersymmetric embeddings of D5-branes wrapped on a three-cycle. However, we will verify that, by wrapping the D5-brane over the three-cycles found in section 3.1, one gets stable non-supersymmetric configurations of the D5-brane probe (see section A.4).

In appendix A we will also analyze the embeddings in which the D5-brane wraps the entire $T^{1,1}$, which correspond to the baryon vertex construction in the KW model. In these configurations the D5-brane worldvolume captures the Ramond-Ramond flux of the background and the worldvolume gauge field cannot be taken to vanish. We will study these embeddings both from the point of view of the Dirac-Born-Infeld action and of the kappa symmetry, and we will conclude that they cannot be supersymmetric.

4.1 D5-branes wrapped on a two-cycle

Let us consider a D5-brane wrapped on a two-cycle of the $T^{1,1}$ space. In order to preserve supersymmetry in a D3-D5 intersection the two branes must share two spatial directions. Accordingly, we will place the D5-brane probe at some constant value of one of the Minkowski coordinates (say x^3) and we will extend it along the radial direction. Following this discussion, let us take the following set of worldvolume coordinates

$$\xi^{\mu} = (x^0, x^1, x^2, r, \theta_1, \phi_1)\,, \tag{4.2}$$

and consider embeddings with x^3 and ψ constant in which

$$\theta_2 = \theta_2(\theta_1, \phi_1)\,, \qquad \phi_2 = \phi_2(\theta_1, \phi_1)\,. \tag{4.3}$$

In this case eq. (4.1) takes the form:

$$\Gamma_{\kappa}\,\epsilon = \frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\;\Gamma_{x^0x^1x^2r}\;\gamma_{\theta_1\phi_1}\,\epsilon^*\,. \tag{4.4}$$

The induced gamma matrices along the θ_1 and ϕ_1 directions are given by the same equations as in the D3-brane embeddings of section 3 (see eq. (3.5)). Denoting by ψ_0 the constant value of the ψ coordinate, we obtain after an straightforward calculation:

$$\frac{6}{L^2}\,\gamma_{\theta_1\phi_1}\,\epsilon^* \;= -ic_1\,\epsilon^* + (c_2 - ic_3)e^{i\psi_0}\,\Gamma_{1\hat{2}}\,\epsilon^* + (\,c_4 - ic_5)\Gamma_{1\hat{3}}\epsilon^* + (c_6 - ic_7)\,e^{i\psi_0}\,\Gamma_{\hat{1}\hat{3}}\,\epsilon^*\,, \tag{4.5}$$

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where the coefficients c_1, c_2 and c_3 are just the same as in the D3-brane (eq. (3.7)) and $c_4, \ldots, c_7$ are given by:

$$\begin{aligned}
c_4 &= \sqrt{\frac{2}{3}}\,[\,\cos\theta_1 + \cos\theta_2\partial_{\phi_1}\phi_2\,]\,,\\
c_5 &= -\sqrt{\frac{2}{3}}\,\sin\theta_1\cos\theta_2\partial_{\theta_1}\phi_2\,,\\
c_6 &= \sqrt{\frac{2}{3}}\,[\,\cos\theta_1 + \cos\theta_2\partial_{\phi_1}\phi_2\,]\,\partial_{\theta_1}\theta_2\,,\\
c_7 &= \sqrt{\frac{2}{3}}\,[\,\cos\theta_1 + \cos\theta_2\partial_{\phi_1}\phi_2\,]\;\partial_{\theta_1}\phi_2\,\sin\theta_2\,.
\end{aligned} \tag{4.6}$$

To implement the $\Gamma_\kappa\epsilon = \epsilon$ condition one must impose some differential (BPS) equations which make some of the c_i coefficients of eq. (4.5) vanish. The remaining terms give rise to an extra projection which must commute with the ones already satisfied by the Killing spinors in order to be compatible with them. By inspecting the different terms on the right-hand side of eq. (4.5), one easily concludes that only the terms with the $\Gamma_{1\hat{2}}$ matrix lead to a projection which commutes with the ones corresponding to the $T^{1,1}$ coset (eq. (2.17)). Therefore, it seems clear that we must require the vanishing of all the coefficients c_i different from c_2 and c_3. Moreover, from the fact that $\Gamma_\kappa^2 = 1$ for any embedding, one easily proves that

$$\sqrt{-g}_{|BPS} = \frac{r^2}{6}\,|c_2 - ic_3|_{BPS}\,. \tag{4.7}$$

Then, if $\delta(\theta_1, \phi_1)$ denotes the phase of $c_2 - ic_3$, it is clear that, if the BPS equations hold, the kappa symmetry condition (2.32) is equivalent to the projection:

$$ie^{i\delta(\theta_1,\phi_1)}\,e^{i\psi_0}\,\Gamma_{x^0x^1x^2r}\,\Gamma_{1\hat{2}}\,\epsilon^* = \epsilon\,. \tag{4.8}$$

We want to translate the above projection (4.8) into an algebraic condition involving a constant matrix acting on a constant spinor. It is rather evident by inspecting eq. (4.8) that this can only be achieved if the phase δ does not depend on the worldvolume angles θ_1 and ϕ_1, which is ensured if $c_2 - ic_3$ is either real or purely imaginary, i.e. when c_2 or c_3 is zero. We will demonstrate later on in this section that by imposing $c_2 = 0$ one does not arrive at a consistent set of equations. Thus, let us consider the case $c_3 = 0$, i.e. let us require that all the coefficients except c_2 vanish:

$$c_1 = c_3 = c_4 = c_5 = c_6 = c_7 = 0\,. \tag{4.9}$$

Notice, first of all, that the condition $c_5 = 0$ implies that $\partial_{\theta_1}\phi_2 = 0$. Substituting this result in the equation $c_3 = 0$ one gets that the other crossed derivative $\partial_{\phi_1}\theta_2$ also vanishes (see eq. (3.7)) and, thus, one must have embeddings of the type:

$$\theta_2 = \theta_2(\theta_1)\,, \qquad \phi_2 = \phi_2(\phi_1)\,. \tag{4.10}$$

For these embeddings c_7 is automatically zero and the conditions $c_4 = c_6 = 0$ give rise to the equation

$$\cos\theta_1 + \cos\theta_2\partial_{\phi_1}\phi_2 = 0\,, \tag{4.11}$$

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while the remaining condition $c_1 = 0$ yields another first-order equation, namely:

$$\sin\theta_1 + \sin\theta_2 \partial_{\theta_1}\theta_2 \partial_{\phi_1}\phi_2 = 0\,. \tag{4.12}$$

Eqs. (4.11) and (4.12) are equivalent to the conditions (4.9) and are the first-order BPS differential equations we were looking for in this case.

Let us now try to find the supersymmetry preserved by the BPS configurations. First of all we notice that

$$\sqrt{-g}_{\;|BPS} = \frac{r^2}{6}\,|c_2|_{\;BPS}\,, \tag{4.13}$$

and thus, the action of Γ_κ on a Killing spinor ϵ when the BPS conditions are satisfied is

$$\Gamma_\kappa\,\epsilon_{\;|BPS} = i\,\mathrm{sign}(c_2)\,e^{i\psi_0}\,\Gamma_{x^0x^1x^2r}\,\Gamma_{1\hat{2}}\,\epsilon^*\,. \tag{4.14}$$

Therefore, we must require that:

$$i\,\mathrm{sign}(c_2)\,e^{i\psi_0}\,\Gamma_{x^0x^1x^2r}\,\Gamma_{1\hat{2}}\,\epsilon^* = \epsilon\,. \tag{4.15}$$

We want to convert eq. (4.15) into an algebraic condition on a constant spinor. With this purpose in mind, let us write the general form of ϵ as the sum of the two types of spinors written in eq. (2.19), namely:

$$\epsilon = r^{-\frac{1}{2}}\,\eta_- + r^{\frac{1}{2}}\,\left(\frac{\bar{x}^3}{L^2}\,\Gamma_{rx^3}\,\eta_- + \eta_+\right) + \frac{r^{\frac{1}{2}}}{L^2}\,x^p\,\Gamma_{rx^p}\,\eta_-\,, \tag{4.16}$$

where $\bar{x}^3$ is the constant value of the coordinate x^3 in the embedding, $\eta_\pm$ are constant spinors satisfying eq. (2.18) and the index p runs over the set $\{0, 1, 2\}$. In eq. (4.16) we have explicitly displayed the dependence of ϵ on the coordinates r and x^p. By substituting eq. (4.16) on both sides of eq. (4.15), one can get the conditions that η_+ and η_- must satisfy.[3] In order to write these conditions, let us define $\mathcal{P}$ as the operator that acts on any spinor ϵ as follows:

$$\mathcal{P}\,\epsilon \equiv \mathrm{sign}(c_2)e^{i\psi_0}\,\Gamma_{rx^3}\,\Gamma_{1\hat{2}}\,\epsilon^*\,. \tag{4.17}$$

Then, eq. (4.15) is equivalent to

$$\begin{aligned} \mathcal{P}\,\eta_- &= \eta_-\,,\\ (1+\mathcal{P}\,)\,\eta_+ &= -\frac{2\bar{x}^3}{L^2}\,\Gamma_{rx^3}\,\eta_-\,. \end{aligned} \tag{4.18}$$

Since $\mathcal{P}^2 = 1$, we can classify the four spinors η_+ according to their $\mathcal{P}$-eigenvalue as:

$$\mathcal{P}\,\eta_+^{(\pm)} = \pm\eta_+^{(\pm)}\,. \tag{4.19}$$

We can now solve the system (4.18) by taking $\eta_- = 0$ and taking η_+ equal to one of the two spinors $\eta_+^{(-)}$ of negative $\mathcal{P}$-eigenvalue. Moreover, there are other two solutions which correspond to taking a spinor $\eta_+^{(+)}$ of positive $\mathcal{P}$-eigenvalue and a spinor η_- related to the

[3] A similar analysis, for the D5-brane configurations in the $AdS_5 \times S^5$ background, was performed in ref. [30]

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former as:

$$\eta_- = \frac{L^2}{\bar{x}^3}\,\Gamma_{rx^3}\,\eta_+^{(+)}\,. \tag{4.20}$$

Notice that, according to the first equation in (4.18), the spinor η_- must have positive $\mathcal{P}$-eigenvalue, in agreement with eq. (4.20). All together this configuration preserves four supersymmetries, i.e. one half of the supersymmetries of the background, as expected for a domain wall.

4.1.1 Integration of the first-order equations

Let us now integrate the BPS equations (4.11) and (4.12). First of all, notice that eq. (4.11) can be written as

$$\frac{\partial\phi_2}{\partial\phi_1} = -\frac{\cos\theta_1}{\cos\theta_2} = \text{constant}\,, \tag{4.21}$$

where we have already taken into account the only way in which eq. (4.11) can be consistent with the dependencies displayed in eq. (4.10). Moreover, by combining eq. (4.12) with eq. (4.11), one can eliminate $\partial_{\phi_1}\phi_2$ and obtain the following equation for $\partial_{\theta_1}\theta_2$:

$$\frac{\partial\theta_2}{\partial\theta_1} = \frac{\tan\theta_1}{\tan\theta_2}\,. \tag{4.22}$$

Eq. (4.22) is easily integrated with the result:

$$\sin\theta_2 = k\sin\theta_1\,, \tag{4.23}$$

where k is a constant. Since $|\cos\theta_2| = \sqrt{1-\sin^2\theta_2} = \sqrt{1-k^2\sin^2\theta_1}$, it follows that

$$\left|\frac{\cos\theta_1}{\cos\theta_2}\right| = \frac{|\cos\theta_1|}{\sqrt{1-k^2\sin^2\theta_1}}\,, \tag{4.24}$$

which is not constant (as required by eq. (4.21)) unless $k=\pm1$. It is easy to conclude that k cannot be equal to -1, since in this case $\sin\theta_2$ would be negative (see eq. (4.23)), which is impossible if $\theta_2\in[0,\pi]$. Thus, $k=1$ and there are two possibilities $\theta_2=\theta_1$ and $\theta_2=\pi-\theta_1$, which correspond respectively to $\cos\theta_2/\cos\theta_1=\pm1$ and $\frac{\partial\phi_2}{\partial\phi_1}=\mp1$. Thus, the two solutions are

$$\begin{aligned} \theta_2 &= \theta_1\,, & \phi_2 &= 2\pi-\phi_1\,,\\ \theta_2 &= \pi-\theta_1\,, & \phi_2 &= \phi_1\,. \end{aligned} \tag{4.25}$$

Notice the similarity of (4.25) and (3.24), although the two solutions are actually very different (see, for example, their different holomorphic structure). Moreover, it is interesting to point out that $c_2 = 2\sin\theta_1$ for the solution with $\theta_2=\theta_1$ while $c_2=-2\sin\theta_1$ when $\theta_2=\pi-\theta_1$. These two solutions correspond to the two possible signs in the projection (4.15). Moreover, the two-cycles (4.25) mix the (θ_1,ϕ_1) and (θ_2,ϕ_2) two-spheres, in agreement with the results of [31].

To finish this subsection, let us discuss the possibility of requiring the vanishing of all c's in eq. (4.5) except for c_3. From the vanishing of c_1, c_4 and c_5 we obtain again eqs. (4.11) and (4.12). Moreover, from $c_2=0$ we obtain a new equation:

$$\partial_{\phi_1}\phi_2 = \frac{\sin\theta_1}{\sin\theta_2}\,\partial_{\theta_1}\theta_2\,. \tag{4.26}$$

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By combining this new equation with eq. (4.11), we obtain:

$$\partial_{\theta_1} \theta_2 = -\frac{\tan\theta_2}{\tan\theta_1}\,. \tag{4.27}$$

It is easy to see that this equation is inconsistent with eq. (4.22) (which follows from eqs. (4.11) and (4.12)). Thus, we conclude that this way of proceeding does not lead to any new solution of the kappa symmetry condition $\Gamma_\kappa\,\epsilon = \epsilon$.

4.1.2 Holomorphic structure

Let us now write the embeddings just found in terms of the holomorphic coordinates $z_1, \cdots, z_4$ of the conifold. From the expressions of the ratios of the z's in terms of the angles (θ_i, ϕ_i) (eq. (2.4)), it is immediate to realize that for the $\theta_1 = \theta_2$ embedding of eq. (4.25) one has:

$$\frac{z_1}{z_3} = \frac{z_4}{z_2} = \frac{\bar z_1}{\bar z_4} = \frac{\bar z_3}{\bar z_2}\,, \tag{4.28}$$

which can be written as a single quadratic equation such as:

$$z_1 \bar z_4 - \bar z_1\, z_3 = 0\,. \tag{4.29}$$

(This embedding also satisfies that $|z_3| = |z_4|$). Notice that the equations found are not holomorphic. This is in correspondence with the fact that the kappa symmetry projector of a D5 involves a complex conjugation of the spinor. Similarly, for the $\theta_2 = \pi - \theta_1$ embedding of eq. (4.25), one has

$$\frac{z_2}{z_3} = \frac{z_4}{z_1} = \frac{\bar z_4}{\bar z_2} = \frac{\bar z_1}{\bar z_3}\,, \tag{4.30}$$

which, again, can be recast as a single quadratic equation, which in this case can be written as:

$$z_1\, \bar z_4 - \bar z_2\, z_4 = 0\,, \tag{4.31}$$

and one has that $|z_1| = |z_2|$ for this solution.

4.1.3 Energy bound

It can be easily verified by explicit calculation that any solution of the type (4.10) of the first-order equations (4.11) and (4.12) is also a solution of the Euler-Lagrange equations derived from the Dirac-Born-Infeld lagrangian density $\mathcal{L} = -\sqrt{-g}$. Actually, for a generic configuration with $\theta_2 = \theta_2(\theta_1)$ and $\phi_2 = \phi_2(\phi_1)$, the hamiltonian density $\mathcal{H} = -\mathcal{L}$ is:

$$\mathcal{H} = \frac{r^2}{6}\,\sqrt{1 + (\partial_{\theta_1}\theta_2)^2}\,\sqrt{\sin^2\theta_1\, + \sin^2\theta_2\,(\partial_{\phi_1}\phi_2)^2 + \frac{2}{3}\,(\,\cos\theta_1 + \cos\theta_2\,\partial_{\phi_1}\phi_2\,)^2}\,. \tag{4.32}$$

Let us now show that $\mathcal{H}$ satisfies a BPS bound of the type $\mathcal{H} \geq |\mathcal{Z}|$, which is saturated precisely when (4.11) and (4.12) are satisfied. First of all, we rewrite $\mathcal{H}$ as $\mathcal{H} = |\mathcal{Z}| + \mathcal{S}$, where

$$\mathcal{Z} = \frac{r^2}{6}\,(\sin\theta_1\,\partial_{\theta_1}\theta_2 - \sin\theta_2\,\partial_{\phi_1}\phi_2\,)\,. \tag{4.33}$$

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It can be straightforwardly proven that $\mathcal{Z}$ can be written as a total derivative, i.e.

$$\mathcal{Z} = \partial_{\theta_1} \mathcal{Z}^{\theta_1} + \partial_{\phi_1} \mathcal{Z}^{\phi_1} \,, \tag{4.34}$$

and it is not difficult to find the explicit expressions of $\mathcal{Z}^{\theta_1}$ and $\mathcal{Z}^{\phi_1}$, namely:

$$\mathcal{Z}^{\theta_1} = \frac{r^2}{6}\,\theta_2 \sin\theta_1 \,, \qquad \mathcal{Z}^{\phi_1} = -\frac{r^2}{6}\,(\phi_2 \sin\theta_2 + \theta_2\phi_1 \cos\theta_1\,)\;. \tag{4.35}$$

Moreover, one can check that $\mathcal{S} \geq 0$ is equivalent to:

$$(\,\sin\theta_1 + \sin\theta_2\,\partial_{\theta_1}\theta_2\partial_{\phi_1}\phi_2\,)^2 + \frac{2}{3}\,\big(1 + (\partial_{\theta_1}\theta_2)^2\big)\,(\,\cos\theta_1 + \cos\theta_2\,\partial_{\phi_1}\phi_2\,)^2 \,\geq\, 0\,, \tag{4.36}$$

which is obviously satisfied and reduces to an equality when the BPS equations (4.11) and (4.12) are satisfied. Clearly, in this case the bound $\mathcal{H} \geq |\mathcal{Z}|$ is saturated. Notice that $\mathcal{Z} \geq 0$ for the BPS embedding of (4.25) with $\theta_2 = \theta_1$, while $\mathcal{Z} \leq 0$ when $\theta_2 = \pi - \theta_1$ and $\phi_2 = \phi_1$.

5. Kappa symmetry for a D7-brane probe

In this section we will try to find supersymmetric embeddings of a D7-brane probe in the $AdS_5 \times T^{1,1}$ geometry. The corresponding kappa symmetry matrix can be obtained from eqs. (2.30) and (2.33), namely:

$$\Gamma_\kappa = -\frac{i}{8!\sqrt{-g}}\,\epsilon^{\mu_1\cdots\mu_8}\,\gamma_{\mu_1\cdots\mu_8}\,. \tag{5.1}$$

The main interest of studying D7-branes in the $AdS_5\times T^{1,1}$ background comes from their use as flavor branes, i.e. as branes whose fluctuations can be identified with dynamical mesons of the corresponding gauge theory. These flavor branes must be spacetime filling, i.e. they must be extended along all the gauge theory directions. Moreover, their worldvolume should include some holographic, non-compact, direction. In this section we will determine some supersymmetric configurations which fulfill these requirements. The study of other embeddings, such as those in which the D7-brane is wrapped on a three-cycle of $T^{1,1}$, is left for appendix A. There exists also the possibility of wrapping supersymmetrically the entire $T^{1,1}$. The corresponding D7-brane embedding preserves two supersymmetries, as shown in appendix A.

5.1 Spacetime filling D7-brane

As explained above, we are interested in D7-brane configurations which extend along the $x^0, \ldots, x^3$ coordinates. Notice that in this case the D7-brane probe and the D3-branes of the background share four dimensions, which is just what is needed for a supersymmetric D3-D7 intersection. Accordingly, let us choose the following set of worldvolume coordinates

$$\xi^\mu = (x^0, \cdots, x^3, \theta_1, \phi_1, \theta_2, \phi_2)\,. \tag{5.2}$$

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The remaining ten-dimensional coordinates r and ψ will be considered as scalars. Actually, we will restrict ourselves to those configurations in which the dependence of r and ψ on the worldvolume coordinates is the following:

$$\psi = \psi(\phi_1, \phi_2)\,, \qquad r = r(\theta_1, \theta_2)\,. \tag{5.3}$$

In this case the kappa symmetry matrix (5.1) takes the form:

$$\Gamma_\kappa = -i\,\frac{h^{-1}}{\sqrt{-g}}\,\Gamma_{x^0\cdots x^3}\,\gamma_{\theta_1\phi_1\theta_2\phi_2}\,, \tag{5.4}$$

where h is the warp factor of eq. (2.5). The induced gamma matrices along the angular coordinates of the worldvolume are:

$$\begin{aligned}
h^{-1/4}\,\gamma_{\theta_1} &= \frac{r}{\sqrt{6}}\,\Gamma_1 + \partial_{\theta_1} r\,\Gamma_r\,,\\
h^{-1/4}\,\gamma_{\phi_1} &= \frac{r}{\sqrt{6}}\,\sin\theta_1\,\Gamma_2 + \frac{r}{3}\,(\cos\theta_1 + \partial_{\phi_1}\psi\,)\,\Gamma_{\hat{3}}\,,\\
h^{-1/4}\,\gamma_{\theta_2} &= \frac{r}{\sqrt{6}}\Big(\cos\psi\,\Gamma_{\hat{1}} + \sin\psi\,\Gamma_{\hat{2}}\Big) + \partial_{\theta_2} r\,\Gamma_r\,,\\
h^{-1/4}\,\gamma_{\phi_2} &= \frac{r}{\sqrt{6}}\,\sin\theta_2\Big(\sin\psi\,\Gamma_{\hat{1}} - \cos\psi\,\Gamma_{\hat{2}}\Big) + \frac{r}{3}\,(\cos\theta_2 + \partial_{\phi_2}\psi\,)\,\Gamma_{\hat{3}}\,.
\end{aligned} \tag{5.5}$$

By inspecting the form of the kappa symmetry matrix in eq. (5.4), one readily concludes that ϵ must be a eigenvector of $\Gamma_* = i\Gamma_{x^0x^1x^2x^3}$. Then, it has to be of the form ϵ_+ (see eq. (2.19)) and we can write

$$\Gamma_\kappa\,\epsilon_+ = -\frac{h^{-1}}{\sqrt{-g}}\,\gamma_{\theta_1\phi_1\theta_2\phi_2}\,\epsilon_+\,. \tag{5.6}$$

Moreover, after taking into account that ϵ_+ has fixed ten-dimensional chirality

$$\Gamma_{x^0x^1x^2x^3}\,\Gamma_{r12\hat{1}\hat{2}\hat{3}}\,\epsilon_+ = -\epsilon_+\,, \tag{5.7}$$

and using eq. (2.17), one can easily verify that

$$\Gamma_{r\hat{3}}\,\epsilon_+ = -i\epsilon_+\,. \tag{5.8}$$

By using these projection conditions and the explicit form of the γ's (eq. (5.5)), one can prove that

$$h^{-1}\,\gamma_{\theta_1\phi_1\theta_2\phi_2}\,\epsilon_+ = d_1\,\epsilon_+ + ie^{-i\psi}\,d_2\,\Gamma_{\hat{1}\hat{3}}\,\epsilon_+ + id_3\,\Gamma_{1\hat{3}}\,\epsilon_+ + e^{-i\psi}\,d_4\,\Gamma_{1\hat{1}}\,\epsilon_+\,, \tag{5.9}$$

where the coefficients d_i are given by:

$$\begin{aligned}
d_1 &= -\frac{r^4}{36}\,\sin\theta_1\sin\theta_2 + \frac{r^3}{18}\Big[\sin\theta_1\partial_{\theta_2} r\,(\cos\theta_2 + \partial_{\phi_2}\psi\,) + \sin\theta_2\partial_{\theta_1} r\,(\cos\theta_1 + \partial_{\phi_1}\psi\,)\Big]\,,\\
d_2 &= \frac{r^3}{6\sqrt{6}}\,\sin\theta_1\Big[\sin\theta_2\partial_{\theta_2} r + \frac{r}{3}\,(\cos\theta_2 + \partial_{\phi_2}\,\psi\,)\Big]\,,\\
d_3 &= \frac{r^3}{6\sqrt{6}}\,\sin\theta_2\Big[\sin\theta_1\partial_{\theta_1} r + \frac{r}{3}\,(\cos\theta_1 + \partial_{\phi_1}\,\psi\,)\Big]\,,\\
d_4 &= \frac{r^3}{18}\Big[\sin\theta_1\;\partial_{\theta_1} r\,(\cos\theta_2 + \partial_{\phi_2}\,\psi\,) - \sin\theta_2\;\partial_{\theta_2} r\,(\cos\theta_1 + \partial_{\phi_1}\,\psi\,)\Big]\,.
\end{aligned} \tag{5.10}$$

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Notice that the terms with d_2, d_3 and d_4 on the right-hand side of eq. (5.9) give rise to projections which are not compatible with those in eq. (2.17). Therefore, in order to have $\Gamma_\kappa\,\epsilon_+ = \epsilon_+$, we impose

$$d_2 = d_3 = d_4 = 0\,. \tag{5.11}$$

The conditions $d_2 = d_3 = 0$ lead to the following differential equations

$$\partial_{\theta_1} r = -\frac{r}{3}\,\frac{\cos\theta_1 + \partial_{\phi_1}\psi}{\sin\theta_1}\,, \qquad \partial_{\theta_2} r = -\frac{r}{3}\,\frac{\cos\theta_2 + \partial_{\phi_2}\psi}{\sin\theta_2}\,, \tag{5.12}$$

which, in turn, imply that $d_4 = 0$. The two differential equations in (5.12) are enough to guarantee the kappa symmetry condition $\Gamma_\kappa\,\epsilon_+ = \epsilon_+$. Actually (see eq. (5.6)), we have to check that $d_1 = -\sqrt{-g}$ when the embedding satisfies eq. (5.12). This fact can be easily verified if one uses that, along the angular directions, the only non-vanishing elements of the induced metric are:

$$\begin{aligned} g_{\theta_i\theta_j} &= h^{1/2}\,\Big[\,\partial_{\theta_i} r\,\partial_{\theta_j} r + \frac{r^2}{6}\,\delta_{ij}\,\Big]\,, \\ g_{\phi_i\phi_j} &= h^{1/2}\,\Big[(\,\cos\theta_i + \partial_{\phi_i}\psi\,)(\,\cos\theta_j + \partial_{\phi_j}\psi\,)\,\frac{r^2}{9} + \sin^2\theta_i\,\frac{r^2}{6}\,\delta_{ij}\,\Big]\,. \end{aligned} \tag{5.13}$$

From the above analysis it is clear that any Killing spinor of the type $\epsilon = \epsilon_+$, with ϵ_+ as in eq. (2.19), satisfies the kappa symmetry condition $\Gamma_\kappa\epsilon = \epsilon$ if the BPS equations (5.12) hold. Then, these embeddings preserve the four ordinary supersymmetries of the background and, thus, they are 1/8 supersymmetric.

5.1.1 Integration of the first-order equations

The BPS equations (5.12) relate the different derivatives of r and ψ. However, notice that according to our ansatz (5.3) the only dependence on ϕ_1 and ϕ_2 in (5.12) comes from the derivatives of ψ. For consistency these derivatives must be constant, i.e.:

$$\partial_{\phi_1}\psi = n_1\,, \qquad \partial_{\phi_2}\psi = n_2\,, \tag{5.14}$$

where n_1 and n_2 are two numbers which label the different solutions of the BPS equations (5.12). Thus, we can write

$$\psi = n_1\phi_1 + n_2\phi_2 + \text{constant}\,. \tag{5.15}$$

We will refer to a solution with given numbers n_1 and n_2 as a (n_1, n_2)-winding embedding. Below we will restrict ourselves to the case in which n_1 and n_2 are integers. Plugging the function (5.15) on the right-hand side of eq. (5.12) one gets an expression for the partial derivatives of $r(\theta_1, \theta_2)$ which is easy to integrate. The result is:

$$r^3 = \frac{C}{\left(\sin\frac{\theta_1}{2}\right)^{n_1+1}\left(\cos\frac{\theta_1}{2}\right)^{1-n_1}\left(\sin\frac{\theta_2}{2}\right)^{n_2+1}\left(\cos\frac{\theta_2}{2}\right)^{1-n_2}}\,, \tag{5.16}$$

where C is a constant of integration.

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Several remarks concerning the function (5.16) are in order. First of all, notice that it is impossible to choose n_1 and n_2 in such a way that r is constant[4] or even that r depends only on one of the angles θ_i. Moreover, $r(\theta_1, \theta_2)$ is invariant under the change $\theta_i \to \pi - \theta_i$ and $n_i \to -n_i$, which means that, in the analysis of the function (5.16), we can restrict ourselves to the case $n_i \geq 0$.

It is also interesting to point out that the function $r(\theta_1, \theta_2)$ always diverges for some particular values of the angles ($\theta_i = 0, \pi$ for $n_i = 0$, $\theta_i = 0$ for $n_i \geq 1$ and $\theta_i = \pi$ for $n_i \leq -1$). Moreover, the probe brane reaches the origin $r = 0$ of the AdS_5 space only when some of the n_i's is such that $|n_i| \geq 2$. When n_1 and n_2 are such that $|n_i| \leq 1$ there is a minimum value r_* of the coordinate r, which is reached at some particular values of the θ_i's. For example, for the $(n_1, n_2) = (0, 0)$ ($(n_1, n_2) = (1, 1)$) solution, $r_*^3 = 4C$ ($r_*^3 = C$), and this value of r is reached when $\theta_1 = \theta_2 = \pi/2$ ($\theta_1 = \theta_2 = \pi$). The existence of this minimal value of r is important when one considers these D7-branes as flavor branes. Indeed, in this case the minimal value of r provides us of an energy scale, which is naturally identified with the mass of the dynamical quarks added to the gauge theory. Taking this fact into account, we would be inclined to think that the above configurations with $n_i = 0, \pm 1$ are the most adequate to be considered as flavor branes for the $AdS_5 \times T^{1,1}$ background.

5.1.2 Holomorphic structure

Let us now prove that the (n_1, n_2)-winding embeddings just discussed can be described by means of a polynomial equation of the type of that written in eq. (3.26), where now the exponents m_i must satisfy that:[5]

$$m_1 + m_2 + m_3 + m_4 \neq 0\,. \tag{5.17}$$

Indeed, from the expression of the phase of both sides of eq. (3.26) in terms of the angular coordinates, one finds that ψ depends on ϕ_1 and ϕ_2 as in eq. (5.15), with n_1 and n_2 being given by:

$$n_1 = \frac{m_1 - m_2 - m_3 + m_4}{m_1 + m_2 + m_3 + m_4}\,, \qquad n_2 = \frac{m_1 - m_2 + m_3 - m_4}{m_1 + m_2 + m_3 + m_4}\,. \tag{5.18}$$

We can confirm the identification (5.18) by extracting $r(\theta_1, \theta_2)$ from the modulus of eq. (3.26). Actually, one can easily demonstrate that the function $r(\theta_1, \theta_2)$ obtained in this way is just given by the right-hand side of eq. (5.16), with the n_1 and n_2 of eq. (5.18). As a further check of these identifications, notice that n_1 and n_2 are left invariant under the transformation (3.29).

The analysis performed above serves to identify some of our solutions with those proposed in the literature as flavor branes for this background. In this sense, notice that the unit-winding case with $n_1 = n_2 = 1$ corresponds to the embedding $z_1 = C$ proposed in ref. [14], while the zero-winding case with $n_1 = n_2 = 0$ is the embedding $z_1 z_2 = C$ first considered in ref. [12] .

[4] Except when the constant C in eq. (5.16) is equal to zero. In this case $r = 0$, $\sqrt{-g}$ also vanishes and the matrix Γ_κ is not well-defined.

[5] For spacetime filling D7-brane embeddings with $m_1 + m_2 + m_3 + m_4 = 0$, see subsection A.6 of appendix A.

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5.1.3 Energy bound

For a D7-brane embedding of the type (5.3), the Dirac-Born-Infeld lagrangian $\mathcal{L} = -\sqrt{-g}$ can be obtained easily from the elements of the induced metric written in eq. (5.13). We have verified that the equations of motion derived from $\mathcal{L}$ are satisfied if the BPS first-order equations (5.12) are fulfilled. Actually, as happened with the other supersymmetric embeddings we have studied, the hamiltonian density $\mathcal{H} = -\mathcal{L}$ satisfies a bound of the type $\mathcal{H} \geq |\mathcal{Z}|$, which is saturated just when the BPS equations (5.12) are verified. In order to prove this statement let us consider, as in our ansatz (5.3), arbitrary functions $\psi(\phi_1, \phi_2)$ and $r = r(\theta_1, \theta_2)$. We define the functions Δ_1 and Δ_2 as follows

$$\Delta_i = -\frac{r}{3}\,\frac{\cos\theta_i + \partial_{\phi_i}\psi}{\sin\theta_i}\,, \qquad (i = 1, 2)\,. \tag{5.19}$$

Notice that the BPS equations (5.12) are just $\partial_{\theta_i} r = \Delta_i$ and the lagrangian $\mathcal{L} = -\sqrt{-g}$ becomes:

$$\mathcal{L} = -r^2 \sin\theta_1 \sin\theta_2 \sqrt{\left((\partial_{\theta_1} r)^2 + (\partial_{\theta_2} r)^2 + \frac{r^2}{6}\right)\left(\Delta_1^2 + \Delta_2^2 + \frac{r^2}{6}\right)}\,. \tag{5.20}$$

Let us now rewrite the hamiltonian density $\mathcal{H} = -\mathcal{L}$ as $\mathcal{H} = |\mathcal{Z}| + \mathcal{S}$, where $\mathcal{Z}$ is given by:

$$\mathcal{Z} = r^2 \sin\theta_1 \sin\theta_2 \left(\frac{r^2}{6} + \partial_{\theta_1} r\,\Delta_1 + \partial_{\theta_2} r\,\Delta_2\right)\,. \tag{5.21}$$

When ψ and r are arbitrary functions of the type (5.3), it is straightforward to prove that $\mathcal{Z}$ is a total divergence:

$$\mathcal{Z} = \partial_{\theta_1} \mathcal{Z}^{\theta_1} + \partial_{\theta_2} \mathcal{Z}^{\theta_2}\,, \tag{5.22}$$

with $\mathcal{Z}^{\theta_i}$ being given by:

$$\mathcal{Z}^{\theta_1} = -\frac{r^4}{12}(\cos\theta_1 + \partial_{\phi_1}\psi)\sin\theta_2\,, \qquad \mathcal{Z}^{\theta_2} = -\frac{r^4}{12}(\cos\theta_2 + \partial_{\phi_2}\psi)\sin\theta_1\,. \tag{5.23}$$

Moreover, the function $\mathcal{S} = \mathcal{H} - |\mathcal{Z}|$ is non-negative. Actually, the condition $\mathcal{S} \geq 0$ is equivalent to:

$$\frac{r^2}{6}\left(\partial_{\theta_1} r - \Delta_1\right)^2 + \frac{r^2}{6}\left(\partial_{\theta_2} r - \Delta_2\right)^2 + \left(\partial_{\theta_1} r\,\Delta_2 - \partial_{\theta_2} r\,\Delta_1\right)^2 \geq 0\,, \tag{5.24}$$

which is obviously always satisfied and reduces to an equality when $\partial_{\theta_i} r = \Delta_i$. Thus, $\mathcal{H} \geq |\mathcal{Z}|$ and, as previously claimed, the BPS conditions (5.12) saturate the bound. It is also clear from the expression of $\mathcal{Z}$ in (5.21) that $\mathcal{Z}_{|BPS} \geq 0$.

6. Summary and conclusions

Let us summarize our main results. We have used kappa symmetry to explore in a systematic way the supersymmetric embeddings of D-brane probes in the $AdS_5 \times T^{1,1}$ geometry. Our method is based on a detailed knowledge of the Killing spinors of the background and

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allows to determine the explicit form of the D-brane embedding, as well as the fraction of supersymmetry preserved by the different configurations. Generically, the supersymmetric embeddings are obtained by integrating a system of first-order BPS differential equations. We have checked in all cases that the solutions of these BPS equations also solve the equations of motion derived from the Dirac-Born-Infeld action of the brane probe. Actually, we have verified that our embeddings saturate an energy bound, as it is expected to occur for a worldvolume soliton.

In the case of a D3-brane we have found a family of three-cycles, which generalizes the ones used in ref. [7] to construct the duals of the dibaryon operators, and we have determined the field content of the dual operator for our cycles. We have also been able to find explicitly the two-cycles over which one must wrap a D5-brane in order to preserve some fraction of supersymmetry, which should correspond to domain walls in the field theory dual. The analysis of spacetime filling configurations of D7-brane probes led us to determine a two-parameter family of supersymmetric embeddings, some of those having the right properties to be considered as flavor branes for the Klebanov-Witten model. Moreover, as shown in appendix A, the D7-brane can wrap completely the $T^{1,1}$ coset and preserve two supersymmetries. We have also shown (see appendix A) that there exist stable non-supersymmetric configurations of a D3-brane (D5–brane) wrapped over a two-cycle (three-cycle). The baryon vertex construction (a D5-brane wrapping the $T^{1,1}$ space) is also studied in appendix A and we conclude that the corresponding configuration is not supersymmetric. All the supersymmetric embeddings we have found can be described by means of a simple polynomial equation in terms of the holomorphic coordinates of the conifold. Notice, however, that our method does not rely on this fact and can be applied to other backgrounds, as in ref. [29], for which the algebraic-geometric techniques are not available. Nevertheless, one might wonder if, for the background studied here, it would not be more appropriate to use the holomorphic coordinates from the beginning, instead of the angular and radial variables that we have employed. To clarify this point, let us recall that the z coordinates are not independent, since they satisfy the constraint (2.2) and, therefore, their use as worldvolume coordinates would be rather cumbersome.

Let us now discuss some possible extensions of our work. We could study the fluctuations of the brane probe around the static configurations we have found. In the case of the D3-brane, this has already been done in ref. [8] for the zero-winding embedding. Moreover, the fluctuations of the spacetime filling D7-branes would allow us to extract the spectrum of dynamical mesons of the theory, as was done in refs. [15, 16, 32, 28] for different cases. Another interesting future research line would be the application of our methodology to other backgrounds. Recall that the starting point of our formalism is a representation of the Killing spinors in a basis in which they become independent of the compact coordinates. For the Klebanov-Strassler background such a representation is obtained in appendix B. It would also be interesting to study the supersymmetric configurations of M2 and M5 brane probes in some backgrounds of eleven-dimensional supergravity. The analogue of the case studied here would be considering a manifold of the type $AdS_4 \times X_7$, where X_7 is a seven-dimensional Einstein space [33]. Work along these lines is in progress and we hope to report on it in a near future.

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Acknowledgments

We are grateful to José D. Edelstein, Sean A. Hartnoll, Carlos Núñez, Angel Paredes, Ruben Portugues and Martin Schvellinger for useful discussions and encouragement. This work is supported in part by MCyT, FEDER and Xunta de Galicia under grant BFM2002-03881 and by the EC Commission under the FP5 grant HPRN-CT-2002-00325.

A. Some other possibilities

In this appendix we explore some possible configurations of brane probes which have not been analyzed in the main text. The first case we will study is a D3-brane extended along one direction in the AdS_5 space and wrapped along two directions of the $T^{1,1}$ coset. We will verify that such "fat" string configurations are not supersymmetric. However, we will show that there exist stable non-supersymmetric embeddings in which the D3-brane wraps the two-cycle found in section 4.1.

We will then consider D5-branes wrapped over two- and three-dimensional submanifolds of the $T^{1,1}$ space. In the first case we will find a supersymmetric configuration not included among those analyzed in section 4 and we will show that one can add flux of the worldvolume gauge field and preserve the same amount of supersymmetry. On the contrary, we will not be able to find any supersymmetric solution for a D5-brane probe wrapped on a three-cycle. Nevertheless, one can wrap the D5-brane on the three-cycles found in section 3.1. We will show that these embeddings are stable non-supersymmetric solutions of the equations of motion. We will also analyze the baryon vertex configuration, which corresponds to a D5-brane wrapping the entire $T^{1,1}$ space. In this case the worldvolume gauge fields cannot be taken to vanish and, despite its similarity with the baryon vertex construction in $AdS_5 \times S^5$, we will conclude that supersymmetry is broken completely.

We will study again the spacetime filling configurations of D7-branes by using an alternative set of worldvolume coordinates which now will include the radial variable r. As a result of this analysis, we will find a class of embeddings in which the D7-brane extends infinitely in r and wraps the three-cycles found in section 3.1. These configurations preserve the same supersymmetries as those found in section 5.1 and can be represented by a polynomial equation of the same type in the holomorphic coordinates of the conifold. Finally, we will show that it is possible to wrap a D7-brane over the full $T^{1,1}$ coset and preserve some supersymmetry.

A.1 D3-branes wrapped on a two-cycle

Let us consider now a D3-brane which shares one direction (say x^1) with the D3-branes of the background and wraps a two-dimensional cycle of the compact $T^{1,1}$ space. Such an object would be one-dimensional from the gauge theory perspective, i.e. it would look like as an inflated string. To describe such a configuration we will take the following set of worldvolume coordinates $\xi^\mu = (x^0, x^1, \theta_1, \phi_1)$ and we will look for embeddings with x^2, x^3, r and ψ constant and $\theta_2 = \theta_2(\theta_1, \phi_1)$ and $\phi_2 = \phi_2(\theta_1, \phi_1)$. In this case the kappa symmetry

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matrix Γ_κ acts on a Killing spinor ϵ as:

$$\Gamma_\kappa\,\epsilon = -\frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\,\Gamma_{x^0x^1}\,\gamma_{\theta_1\phi_1}\,\epsilon\,. \tag{A.1}$$

The induced matrices γ_{θ_1} and γ_{ϕ_1} are the same as those written in eq. (3.5) and the action of $\gamma_{\theta_1\phi_1}$ on ϵ can be obtained as in eq. (4.5):

$$\frac{6}{L^2}\,\gamma_{\theta_1\phi_1}\,\epsilon = ic_1\,\epsilon + (c_2+ic_3)e^{-i\psi_0}\,\Gamma_{1\hat{2}}\,\epsilon + (c_4+ic_5)\Gamma_{1\hat{3}}\epsilon + (c_6+ic_7)\,e^{-i\psi_0}\,\Gamma_{\hat{1}\hat{3}}\,\epsilon\,, \tag{A.2}$$

where c_1, c_2 and c_3 are given in eq. (3.7), while $c_4,\cdots,c_7$ are displayed in eq. (4.6).

The only projection compatible with the $T^{1,1}$ conditions (2.17) is the one originated from the first term on the right-hand side of eq. (A.2). Accordingly, let us require that $c_2=c_3=\cdots=c_7=0$. From $c_2=c_3=0$ we obtain the Cauchy-Riemann equations (3.9). Moreover, from $c_5=0$ it follows that $\phi_2=\phi_2(\phi_1)$ which ensures that c_7 vanishes, while by using this result on the second equation in (3.9) one concludes that $\theta_2=\theta_2(\theta_1)$. On the other hand, the conditions $c_4=c_6=0$ lead to eq. (4.21), i.e. to $\partial_{\phi_1}\phi_2 = -\cos\theta_1/\cos\theta_2 =$ constant. By combining this result with the first equation in (3.9), one gets the same result as in eq. (4.27), namely $\partial_{\theta_1}\theta_2 = -\tan\theta_2/\tan\theta_1$, which can be easily integrated with the result $\sin\theta_2\sin\theta_1 = k$, with k being a constant which necessarily must satisfy $k\le 1$. It is not difficult to show that this solution is inconsistent since, for example, $\cos\theta_1/\cos\theta_2$ cannot be constant.

If we give up the requirement of supersymmetry, it is not difficult to find stable solutions of the equations of motion. Indeed, apart from an irrelevant global factor, the lagrangian of the D3-brane probe considered here is given by the same expression as the one obtained in section 4.1 for a D5-brane wrapping a two-cycle. It follows from our analysis of section 4.1 that any solution of eqs. (4.11) and (4.12) also solves the Euler-Lagrange equations of motion. Recall that the solutions of eqs. (4.11) and (4.12) have been written in eq. (4.25). Thus, the functions (4.25) are also a solution of the D3-brane equations of motion. Moreover, from the equivalence between the D3- and D5-brane hamiltonians and the results of section 4.1.3, one can also establish a bound for the D3-brane energy, which is saturated for the configurations (4.25). This fact ensures that these embeddings, although they break supersymmetry completely, are stable.

A.2 More D5-branes wrapped on a two-cycle

Let us consider a D5-brane wrapped on a two-cycle of the $T^{1,1}$ space and let us take the following set of worldvolume coordinates $\xi^\mu = (x^0,x^1,x^2,r,\theta_1,\theta_2)$. We shall consider embeddings with x^3 and ψ constant in which $\phi_1=\phi_1(\theta_1,\theta_2)\,,\phi_2=\phi_2(\theta_1,\theta_2)$. Particularizing eq. (4.1) to this case, we arrive at:

$$\Gamma_\kappa\,\epsilon = \frac{i}{\sqrt{-g}}\,\frac{r^2}{L^2}\,\Gamma_{x^0x^1x^2r}\,\gamma_{\theta_1\theta_2}\,\epsilon^*\,. \tag{A.3}$$

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Denoting by ψ_0 the constant value of ψ, we obtain the following value of the induced gamma matrices:

$$\begin{aligned}
\gamma_{\theta_1} &= \frac{L}{\sqrt{6}}\left[\Gamma_1+\sin\theta_1\partial_{\theta_1}\phi_1\Gamma_2+\sin\theta_2\partial_{\theta_1}\phi_2(\sin\psi_0\Gamma_{\hat{1}}-\cos\psi_0\Gamma_{\hat{2}})\right]+\\
&\quad+\frac{L}{3}\left[\cos\theta_1\partial_{\theta_1}\phi_1+\cos\theta_2\partial_{\theta_1}\phi_2\right]\Gamma_{\hat{3}}\,,\\
\gamma_{\theta_2} &= \frac{L}{\sqrt{6}}\left[(\cos\psi_0+\sin\psi_0\sin\theta_2\partial_{\theta_2}\phi_2)\Gamma_{\hat{1}}+(\sin\psi_0-\cos\psi_0\sin\theta_2\partial_{\theta_2}\phi_2)\Gamma_{\hat{2}}+\sin\theta_1\partial_{\theta_2}\phi_1\Gamma_2\right]+\\
&\quad+\frac{L}{3}\left[\cos\theta_1\partial_{\theta_2}\phi_1+\cos\theta_2\partial_{\theta_2}\phi_2\right]\Gamma_{\hat{3}}\,. \qquad\qquad (A.4)
\end{aligned}$$

After an straightforward calculation, one can demonstrate that:

$$\gamma_{\theta_1\theta_2}\,\epsilon^* = if_1\,\epsilon^*+(f_2+if_3)\,e^{i\psi_0}\,\Gamma_{1\hat{1}}\,\epsilon^*+(f_4+if_5)\Gamma_{1\hat{3}}\epsilon^*+(f_6+if_7)\,e^{i\psi_0}\,\Gamma_{\hat{1}\hat{3}}\,\epsilon^*\,, \qquad (A.5)$$

with $f_1,\ldots,f_7$ being given by:

$$\begin{aligned}
f_1 &= \frac{L^2}{6}\left[\sin\theta_2\partial_{\theta_1}\phi_2-\sin\theta_1\partial_{\theta_2}\phi_1\right],\\
f_2 &= \frac{L^2}{6}\left[1+\sin\theta_1\sin\theta_2\left(\partial_{\theta_2}\phi_1\partial_{\theta_1}\phi_2-\partial_{\theta_1}\phi_1\partial_{\theta_2}\phi_2\right)\right],\\
f_3 &= -\frac{L^2}{6}\left[\sin\theta_1\partial_{\theta_1}\phi_1+\sin\theta_2\partial_{\theta_2}\phi_2\right],\\
f_4 &= \frac{L^2}{3\sqrt{6}}\Lambda_2\,,\\
f_5 &= \frac{L^2}{3\sqrt{6}}\sin\theta_1\left[\partial_{\theta_2}\phi_1\Lambda_1-\partial_{\theta_1}\phi_1\Lambda_2\right],\\
f_6 &= -\frac{L^2}{3\sqrt{6}}\Lambda_1\,,\\
f_7 &= \frac{L^2}{3\sqrt{6}}\sin\theta_2\left[\partial_{\theta_2}\phi_2\Lambda_1-\partial_{\theta_1}\phi_2\Lambda_2\right], \qquad\qquad (A.6)
\end{aligned}$$

and we have introduced the quantities Λ_1 and Λ_2, defined as:

$$\Lambda_i\equiv\cos\theta_1\partial_{\theta_i}\phi_1+\cos\theta_2\partial_{\theta_i}\phi_2\,,\qquad (i=1,2)\,. \qquad (A.7)$$

In order to have a projection compatible with that of eq. (2.17), we shall require

$$f_1=f_4=f_5=f_6=f_7=0\,. \qquad (A.8)$$

Taking into account that the vanishing of f_4, f_5, f_6 and f_7 is equivalent to the conditions $\Lambda_1=\Lambda_2=0$, we arrive at the following system of first-order differential equations:

$$\begin{aligned}
\sin\theta_1\partial_{\theta_2}\phi_1 &= \sin\theta_2\partial_{\theta_1}\phi_2\,,\\
\cos\theta_1\partial_{\theta_i}\phi_1 &= -\cos\theta_2\partial_{\theta_i}\phi_2\,,\qquad (i=1,2)\,. \qquad (A.9)
\end{aligned}$$

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A solution of (A.9) can be found by the method of separation of variables. The result is:

$$\begin{aligned}\phi_1 &= A\,\left(\frac{\cos\theta_2}{\cos\theta_1}\right)^{\alpha} + \phi_1^0\,,\\ \phi_2 &= \frac{\alpha}{1-\alpha}\,A\,\left(\frac{\cos\theta_1}{\cos\theta_2}\right)^{1-\alpha} + \phi_2^0\,,\end{aligned} \tag{A.10}$$

where A, α, ϕ_1^0 and ϕ_2^0 are constants. Notice that, when the constant A (or α) vanishes, the above solution reduces to that in which ϕ_1 and ϕ_2 are both constant. Moreover, in agreement with our discussion of section 4, the phase of $f_2 + if_3$ should be constant. By plugging the solution written in (A.10) into the expressions of f_2 and f_3 (eq. (A.6)), it is easy to convince oneself that this only happens if $A = 0$, i.e. for the solution with ϕ_1 and ϕ_2 constant. Therefore, this is the only admissible solution. Actually, one can easily check that it satisfies the equations of motion and preserves the same four supersymmetries as in eq. (4.15) with $\text{sign}(c_2)$ changed by $\text{sign}(f_2) = +1$.

A.3 D5-branes wrapped on a two-cycle with flux

We want now to analyze the effect of adding flux of the worldvolume gauge field to the configurations studied in section 4. Accordingly, let us switch on q units of worldvolume flux along the angular directions θ_1 and ϕ_1. The corresponding field strength is

$$F_{\theta_1\phi_1} = q\sin\theta_1\,. \tag{A.11}$$

When $q \neq 0$, the Wess-Zumino term of the Dirac-Born-Infeld action does not vanish anymore. Indeed, the pullback $P[C^{(4)}]$ of the Ramond-Ramond four-form potential has a component of the form

$$P[C^{(4)}]_{x^0x^1x^2r} = h^{-1}\,\frac{dx^3}{dr}\,, \tag{A.12}$$

and the corresponding term in the action is:

$$\mathcal{L}_{WZ} = -q\,h^{-1}\,x'\,\sin\theta_1\,, \tag{A.13}$$

where we have denoted x^3 simply as x and the prime means derivation with respect to r. The existence of this term implies that x^3 cannot be taken to be independent of r if q is non-vanishing. In order to find this dependence, let us write the lagrangian density for the same angular embedding as in the zero-flux case (eq. (4.25)) and for an arbitrary function $x(r)$. One gets:

$$\mathcal{L} = \sin\theta_1\left[-\left(\frac{L^4}{9} + q^2\right)^{1/2} h^{-1/2}\,(1 + h^{-1}\,(x')^2)^{1/2} - q\,h^{-1}\,x'\right]. \tag{A.14}$$

The equation of motion for x derived from $\mathcal{L}$ implies

$$\left(\frac{L^4}{9} + q^2\right)^{\frac{1}{2}}\,\frac{h^{-3/2}\,x'}{(1 + h^{-1}\,(x')^2\,)^{1/2}} + q\,h^{-1} = \text{constant}\,. \tag{A.15}$$

Let us take the constant on the right-hand side of eq. (A.15) equal to zero. Then, equation (A.15) implies

$$(x')^2 = \frac{9q^2}{r^4} \, . \tag{A.16}$$

Let us consider the solution of the equation of motion with

$$x' = -\frac{3q}{r^2} \, . \tag{A.17}$$

which, after integration, becomes:

$$x = \bar{x}^3 + \frac{3q}{r} \, , \tag{A.18}$$

with $\bar{x}^3$ constant. We are now going to prove that this solution of the equations of motion is supersymmetric. Notice that the expression of the kappa symmetry matrix Γ_κ differs from that written in eq. (2.30), due to the non-zero value of the worldvolume gauge field. Actually, from the general expression of Γ_κ given in ref. [18], one can easily prove that, for the case at hand, one has

$$\Gamma_\kappa \, \epsilon = \frac{i}{\sqrt{-\det(g+F)}} \, \frac{r^3}{L^3} \, \Gamma_{x^0x^1x^2} \left[\gamma_r \, \gamma_{\theta_1\phi_1} \, \epsilon^* \; - \; \gamma_r \, F_{\theta_1\phi_1} \, \epsilon \, \right] . \tag{A.19}$$

Notice that γ_r is given by:

$$\gamma_r = \frac{L}{r} \left(\Gamma_r + \frac{r^2}{L^2} \, x' \, \Gamma_{x^3} \right) , \tag{A.20}$$

and for the angular embeddings of eq. (4.25) we have proved in section 4 that

$$\gamma_{\theta_1\phi_1} \, \epsilon^* = \text{sign}(c_2) \, \frac{L^2}{3} \sin\theta_1 \, e^{i\psi_0} \, \Gamma_{1\hat{2}} \, \epsilon^* \, . \tag{A.21}$$

Taking x' and $F_{\theta_1\phi_1}$ as given in eqs. (A.17) and (A.11) respectively, one gets:

$$\Gamma_\kappa \, \epsilon = \frac{i}{1+\frac{9q^2}{L^4}} \Gamma_{x^0x^1x^2r} \left[\text{sign}(c_2) e^{i\psi_0} \Gamma_{1\hat{2}} \, \epsilon^* - \frac{3q}{L^2} \text{sign}(c_2) e^{i\psi_0} \Gamma_{rx^3} \Gamma_{1\hat{2}} \epsilon^* - \frac{3q}{L^2} \epsilon + \frac{9q^2}{L^4} \, \Gamma_{rx^3} \, \epsilon \right] . \tag{A.22}$$

Moreover, by plugging in eq. (2.15) the explicit dependence of x on r (eq. (A.18)), one gets that the Killing spinor ϵ evaluated on the worldvolume can be written as:

$$\epsilon = r^{-1/2} \left(1 + \frac{3q}{L^2} \, \Gamma_{rx^3} \right) \eta_- + r^{1/2} \left(\frac{\bar{x}^3}{L^2} \, \Gamma_{rx^3} \, \eta_- + \eta_+ \right) \; + \; \frac{r^{\frac{1}{2}}}{L^2} \, x^p \, \Gamma_{rx^p} \, \eta_- \, , \tag{A.23}$$

where the constant spinors $\eta_\pm$ have been defined in eq. (2.18). By using the expression of ϵ given in eq. (A.23) in the equation $\Gamma_\kappa \epsilon = \epsilon$, one finds that, remarkably, the kappa symmetry condition is verified if η_+ and η_- satisfy the system (4.18). Therefore, this configuration preserves the same four supersymmetries as in the zero flux case studied in section 4.

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A.4 D5-branes wrapped on a three-cycle

Let us now explore the possibility of having D5-brane probes wrapping a three-cycle. To represent these configurations, let us proceed as in section 3 and take the following set of worldvolume coordinates $\xi^\mu = (x^0, x^1, x^2, \theta_1, \phi_1, \psi)$, with $\theta_2 = \theta_2(\theta_1, \phi_1)$, $\phi_2 = \phi_2(\theta_1, \phi_1)$ and the remaining two other coordinates x^3 and r being constant. From the general expression (4.1), we obtain:

$$\Gamma_\kappa\, \epsilon = \frac{i}{\sqrt{-g}}\, \frac{r^3}{L^3}\, \Gamma_{x^0x^1x^2}\, \gamma_{\theta_1\phi_1\psi}\, \epsilon^*\,. \tag{A.24}$$

The value of $\gamma_{\theta_1\phi_1\psi}\, \epsilon^*$ can be obtained by taking the complex conjugate of eq. (3.6), namely:

$$\frac{18}{L^3}\, \gamma_{\theta_1\phi_1\psi}\, \epsilon^* = -ic_1\Gamma_{\hat{3}}\epsilon^* + (c_2 - ic_3)\, e^{i\psi}\, \Gamma_{1\hat{2}\hat{3}}\, \epsilon^*\,. \tag{A.25}$$

(The c_i coefficients are given in eq. (3.7)). The only terms in (A.25) which could give rise to a projector compatible with the $T^{1,1}$ conditions (2.17) are the ones containing the matrix $\Gamma_{1\hat{2}\hat{3}}$. Thus, we could try to impose the equation $c_1 = 0$. Notice, however, that the resulting kappa symmetry projector is always going to depend on the worldvolume coordinate ψ, due to the $e^{i\psi}$ factor of the right-hand side of eq. (A.25). We would have in this case a different projector for every point of the worldvolume, which is, clearly, unacceptable.

As an alternative solution to the problem just found, we could try to use a set of worldvolume coordinates which does not include the coordinate ψ, i.e. we will consider ψ as a constant scalar. After some calculations, one can convince oneself that there is no consistent solution also in this approach. Therefore, we are led to conclude that these types of configurations are not supersymmetric.

As in section A.1, it is not difficult to find stable non-supersymmetric embeddings of D5-branes wrapped on three-cycles. Indeed, for the election of worldvolume coordinates and the ansatz for the scalar fields considered above, the lagrangian density of the D5-brane probe is, up to irrelevant factors, the same as the one obtained in section 3.1 for a D3-brane wrapping a three-cycle. We know from the results of section 3.1 that the first-order equations (3.9) imply the fulfillment of the equations of motion. Thus, any solution of (3.9) gives a possible embedding of a D5-brane which wraps a three-cycle. Since, as in section 3.1.3, these embeddings saturate an energy bound, they are stable, despite of the fact that they do not preserve any supersymmetry.

A.5 The baryon vertex

According to Witten's original argument [3], the baryon vertex for the Klebanov-Witten model must correspond to a D5-brane wrapped over the entire $T^{1,1}$ space. Indeed, in this case the D5-brane captures the flux of the Ramond-Ramond five-form $F^{(5)}$, which acts as a source for the worldvolume electric field and, as a consequence, one must have fundamental strings emanating from the D5-brane. The Dirac-Born-Infeld action must now necessarily include a worldvolume gauge field F and a Wess-Zumino term. It takes the form:

$$S = -T_5 \int d^6\xi\, \sqrt{-\det(g+F)} - T_5 \int d^6\xi\, A \wedge F^{(5)}\,, \tag{A.26}$$

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where A is the one-form potential for F, i.e. $F = dA$ and T_5 is the tension of the D5-brane. For simplicity we are going to take from now on the string coupling g_s equal to one. Moreover, to obtain easily the contribution of the Wess-Zumino term in (A.26), it is useful to rewrite the five-form $F^{(5)}$ of eq. (2.5) as:

$$F^{(5)} = \frac{4L^4}{108} \sin\theta_1 \sin\theta_2\, d\theta_1 \wedge d\phi_1 \wedge d\theta_2 \wedge d\phi_2 \wedge d\psi + \text{Hodge dual}\,. \tag{A.27}$$

We shall take the following set of worldvolume coordinates

$$\xi^\mu = (x^0, \theta_1, \phi_1, \theta_2, \phi_2, \psi)\,. \tag{A.28}$$

For this election of the ξ^μ's, it is clear from the expression of $F^{(5)}$ displayed in eq. (A.27) that the one-form potential A must have the form $A = A_0 dx^0$. Actually, we will adopt an ansatz in which the radial coordinate r and the gauge potential A_0 depend only on the angle θ_1, i.e.:

$$r = r(\theta_1)\,, \qquad A_0 = A_0(\theta_1)\,. \tag{A.29}$$

Notice that, within this ansatz, a worldvolume electric field $F_{x^0\theta_1} = -\partial_{\theta_1} A_0$ is switched on along the θ_1 direction. The action (A.26) for such a configuration can be written as:

$$S = \frac{T_5 L^4}{108}\, V_4 \int dx^0 d\theta_1\, \mathcal{L}_{\text{eff}}\,, \tag{A.30}$$

where V_4 is

$$V_4 = \int d\phi_1 d\theta_2 d\phi_2 d\psi \sin\theta_2 = 32\pi^3\,, \tag{A.31}$$

and the effective lagrangian density $\mathcal{L}_{\text{eff}}$ is given by:

$$\mathcal{L}_{\text{eff}} = -\sin\theta_1 \sqrt{\frac{r^2}{6} + (r')^2 - (F_{x^0\theta_1})^2} - 4\sin\theta_1\, A_0\,. \tag{A.32}$$

Let us now introduce the displacement field, defined as:

$$D(\theta_1) \equiv \frac{\partial \mathcal{L}_{\text{eff}}}{\partial F_{x^0\theta_1}} = \sin\theta_1 \frac{F_{x^0\theta_1}}{\sqrt{\frac{r^2}{6} + (r')^2 - (F_{x^0\theta_1})^2}}\,. \tag{A.33}$$

The equation of motion for A_0 derived from $\mathcal{L}_{eff}$ gives the Gauss' law for $D(\theta_1)$:

$$D'(\theta_1) = 4\sin\theta_1\,, \tag{A.34}$$

which can be immediately integrated, namely:

$$D(\theta_1) = -4\cos\theta_1 + \text{constant}\,. \tag{A.35}$$

Integrating by parts the Wess-Zumino term in $\mathcal{L}_{\text{eff}}$ we can substitute the gauge potential A_0 by its canonically conjugate momentum D. This is of course equivalent to performing a Legendre transformation. The resulting hamiltonian is:

$$H = \frac{T_5 L^4}{108}\, V_4 \int d\theta_1\, \mathcal{H}\,, \tag{A.36}$$

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where $\mathcal{H}$ is given by:

$$\mathcal{H} = \sin\theta_1 \sqrt{\frac{r^2}{6} + (r')^2 \, - (F_{x^0\theta_1})^2} + D(\theta_1)\, F_{x^0\theta_1} \,. \tag{A.37}$$

Since $F_{x^0\theta_1}$ and $D(\theta_1)$ are related by (A.33), we can eliminate in $\mathcal{H}$ the electric field $F_{x^0\theta_1}$ in favor of the displacement $D(\theta_1)$, whose explicit expression is known from the integration of the Gauss' law (see eq. (A.35)). Indeed, by inverting eq. (A.33), we get:

$$F_{x^0\theta_1} = \sqrt{\frac{\frac{r^2}{6} + (r')^2}{D(\theta_1)^2 + \sin^2\theta_1}}\, D(\theta_1)\,, \tag{A.38}$$

and, by using this relation, the hamiltonian density becomes:

$$\mathcal{H} = \sqrt{D(\theta_1)^2 + \sin^2\theta_1}\,\sqrt{\frac{r^2}{6} + (r')^2}\,. \tag{A.39}$$

The solutions of the equations of motion are the functions $r(\theta)$ which minimize the above energy functional. Actually, the Euler-Lagrange equations derived from $\mathcal{H}$ are rather involved and we will not try to find directly an analytical solution. Instead, what we have tried is to apply the method used in ref. [34] for the analysis of the baryon vertex in the $AdS_5 \times S^5$ background. In ref. [34], a first-order differential equation for $r(\theta)$ was found. The solutions of this equation also solve the second-order Euler-Lagrange equation and saturate an energy bound. Despite the similarity of our system with the one studied in ref. [34], we have not been able to find the first-order equation and the corresponding energy bound. This fact suggests that the baryon vertex configurations in the $AdS_5 \times T^{1,1}$ geometry are not supersymmetric. To confirm this result, let us consider the kappa symmetry equation for this case. Using the general expression of Γ_κ written in ref. [18] one obtains

$$\Gamma_\kappa\,\epsilon = -\frac{i}{\sqrt{-\det(\,g+F\,)}}\left[\frac{r}{L}\,\Gamma_{x^0}\,\gamma_{\theta_1\phi_1\theta_2\phi_2\psi}\;\epsilon^* - F_{x^0\theta_1}\,\gamma_{\phi_1\theta_2\phi_2\psi}\;\epsilon\,\right]. \tag{A.40}$$

Moreover, since

$$\begin{aligned}
\gamma_{\theta_1\phi_1\theta_2\phi_2\psi}\;\epsilon^* \;&=\; -\frac{L^5}{108}\,\sin\theta_1\sin\theta_2\left(\Gamma_{\hat{3}} + \sqrt{6}\,\frac{r'}{r}\,\Gamma_{r1\hat{3}}\right)\epsilon^*\,,\\
\gamma_{\phi_1\theta_2\phi_2\psi}\;\epsilon \;&=\; -\frac{L^4}{18\sqrt{6}}\,\sin\theta_1\sin\theta_2\,\Gamma_{1\hat{3}}\,\epsilon\,,
\end{aligned} \tag{A.41}$$

we have

$$\Gamma_\kappa\epsilon = \frac{i}{\sqrt{-\det(g+F)}}L^4\sin\theta_1\sin\theta_2\left[\frac{r}{108}\Gamma_{x^0}\Gamma_{\hat{3}}\epsilon^* + \frac{1}{18\sqrt{6}}\Big(r'\Gamma_{x^0r1\hat{3}}\epsilon^* - F_{x^0\theta_1}\Gamma_{1\hat{3}}\epsilon\Big)\right]. \tag{A.42}$$

The kappa symmetry analysis of the baryon vertex in the $AdS_5 \times S^5$ geometry was performed in ref. [35]. The general strategy followed in [35] to solve the $\Gamma_\kappa\,\epsilon = \epsilon$ equation was to try to impose an extra projection on the spinor in such a way that the contributions of

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the worldvolume gauge field $F_{x^0\theta_1}$ and of r' cancel with each other. In our case it is clear that, by requiring that $\Gamma_{x^0r}\epsilon^* = -\epsilon$ the last two terms on the right-hand side of eq. (A.42) cancel with each other if $F_{x^0\theta_1} = -r'$. Notice also that the condition $\Gamma_{x^0r}\epsilon^* = -\epsilon$ corresponds to having fundamental strings along the radial direction, which is just what we expect for a baryon vertex configuration. Moreover, as can be checked by using the fact that ϵ has fixed ten-dimensional chirality, this extra projection is equivalent to require that $i\Gamma_{x^0\hat{3}}\epsilon^* = \epsilon$, which in turn is essential to satisfy the $\Gamma_\kappa\,\epsilon = \epsilon$ equation. However, the fundamental string projection $\Gamma_{x^0r}\epsilon^* = -\epsilon$ is not consistent with the conditions (2.17) satisfied by the Killing spinors and, thus, it cannot be imposed to the ϵ's. Therefore, as suspected, we are not able to solve the $\Gamma_\kappa\,\epsilon = \epsilon$ equation and we are led to conclude that the baryon vertex configuration breaks supersymmetry completely. Actually, from this incompatibility argument we see that this conclusion is more general than the particular ansatz (A.29) that we have chosen.

A.6 More spacetime filling D7-branes

The election of worldvolume coordinates for the D7-brane probe that we have made in section 5.1 might seem arbitrary. For this reason we will adopt here a different point of view and take the following set of worldvolume coordinates

$$\xi^\mu = (x^0, x^1, x^2, x^3, \theta_1, \phi_1, \psi, r)\,, \tag{A.43}$$

and we will consider configurations in which $\theta_2 = \theta_2(\theta_1, \phi_1)$, $\phi_2 = \phi_2(\theta_1, \phi_1)$. In this case, the kappa symmetry matrix Γ_κ takes the form:

$$\Gamma_\kappa = -\frac{i}{\sqrt{-g}}\,\frac{r^4}{L^4}\,\Gamma_{x^0x^1x^2x^3}\,\gamma_{\theta_1\phi_1\psi r}\,. \tag{A.44}$$

Acting on a spinor ϵ_+ such that $\Gamma_*\epsilon_+ = \epsilon_+$ (with Γ_* defined in eq. (2.14)), and using eq. (5.8) and eq. (3.6), one can demonstrate that:

$$\frac{18r}{L^4}\,\gamma_{\theta_1\phi_1\psi r}\,\epsilon_+ = -c_1\epsilon_+ + i(c_2 + ic_3)\,e^{-i\psi}\,\Gamma_{1\hat{2}}\,\epsilon_+\,, \tag{A.45}$$

where c_1, c_2 and c_3 are given in eq. (3.7). The compatibility of the kappa symmetry projection and eq. (2.17) requires imposing $c_2 = c_3 = 0$, i.e. the first-order differential equations (3.9), whose general solution was found in section 3.1.1 (eq. (3.20)). Since $\sqrt{-g}_{|BPS} = \frac{r^3}{18}\,c_{1_{|BPS}}$, one concludes that $\Gamma_{\kappa_{|BPS}}\,\epsilon_+ = \epsilon_+$, i.e. the embeddings which satisfy eq. (3.9) preserve four supersymmetries. Moreover, it is also immediate that the results of section 3.1.3 carry over to this case and, as a consequence, one can establish a bound for the energy, which is saturated by the solutions of (3.9). In particular, the m-winding solutions (3.22) and (3.23) can be represented by the polynomial equation (3.26) (as in section 5.1), but now subjected to the conditions (3.27).

Notice that, in the BPS configurations just described, the D7-brane worldvolume extends infinitely in r and wraps a compact three-cycle of the $T^{1,1}$ space. In particular, the D7-brane reaches the origin at $r = 0$, contrary to what happens to some of the embeddings studied in section 5.1. This type of embeddings have been studied in refs. [14, 36].

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A.7 D7-branes wrapped on $T^{1,1}$

A D7-brane can wrap the entire $T^{1,1}$ space and extend along two other spatial directions of AdS_5. In order to find out if supersymmetry can be preserved in this setup, let us choose the following set of worldvolume coordinates: $\xi^\mu = (x^0, x^1, r, \theta_1, \phi_1, \theta_2, \phi_2, \psi)$. The remaining cartesian coordinates x^2 and x^3 are scalars which, in principle, can depend on the ξ^μ's. First of all, let us assume that x^2 and x^3 are constant and let us study whether or not this embedding is supersymmetric. The general equation (2.30) for this case becomes:

$$\Gamma_\kappa = -\frac{i}{\sqrt{-g}}\,\gamma_{x^0x^1r\theta_1\phi_1\theta_2\phi_2\psi}\,. \tag{A.46}$$

For the configuration that we are considering, one can demonstrate that Γ_κ acts on the Killing spinors as:

$$\Gamma_\kappa\,\epsilon = i\Gamma_{x^0x^1r\hat{3}}\,\epsilon\,. \tag{A.47}$$

This equation is clearly solved for a spinor $\epsilon_+ = r^{1/2}\,\eta_+$ (see eq. (2.19)), where η_+ satisfies the extra projection

$$i\Gamma_{x^0x^1r\hat{3}}\,\eta_+ = \eta_+\,. \tag{A.48}$$

Thus, this configuration preserves two supersymmetries. Moreover, by allowing x^2 and x^3 to depend on the worldvolume coordinates, one can convince oneself that x^2 and x^3 must be necessarily constant in order to preserve some supersymmetry and, thus, the only supersymmetric embeddings are just the ones studied above.

B. Supersymmetry of the Klebanov-Strassler solution

The Killing spinors of a supergravity background are obtained by requiring the vanishing of the supersymmetry variations of the fermionic fields of the theory. For type IIB Sugra with constant dilaton these supersymmetry variations are [37]:

$$\begin{aligned}
\delta\lambda &= -\frac{i}{24}\,F_{\mu_1\mu_2\mu_3}\,\Gamma^{\mu_1\mu_2\mu_3}\,\epsilon\,,\\
\delta\psi_\mu &= D_\mu\,\epsilon + \frac{i}{1920}\,F^{(5)}_{\mu_1\cdots\mu_5}\,\Gamma^{\mu_1\cdots\mu_5}\Gamma_\mu\epsilon + \frac{1}{96}F_{\mu_1\mu_2\mu_3}\,(\Gamma_\mu^{\ \mu_1\mu_2\mu_3} - 9\delta_\mu^{\mu_1}\,\Gamma^{\mu_2\mu_3})\,\epsilon^*\,,
\end{aligned} \tag{B.1}$$

where $\lambda(\psi)$ is the dilatino (gravitino) and F is the following complex combination of the Neveu-Schwarz-Neveu-Schwarz (H) and Ramond-Ramond ($F^{(3)}$) three-forms:

$$F_{\mu_1\mu_2\mu_3} = g_s^{-1/2}\,H_{\mu_1\mu_2\mu_3} + ig_s^{1/2}\,F^{(3)}_{\mu_1\mu_2\mu_3}\,. \tag{B.2}$$

In what follows we will find the spinors ϵ which make that $\delta\lambda = \delta\psi_\mu = 0$ for several backgrounds of type IIB supergravity. First of all we will consider the space obtained by performing a direct multiplication of a four-dimensional Minkowski space and a deformed conifold. This geometry solves the equations of motion of type IIB supergravity without forms and we will be able to find a simplified expression for its Killing spinors. Then, we

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will add D3-branes, that warp the geometry and introduce a non-vanishing value for the Ramond-Ramond five-form $F^{(5)}$, and the result will be reflected on the Killing spinors as an extra projection to be satisfied by them and the multiplication of ϵ by a power of the warp factor. The Klebanov-Strassler solution is obtained by adding three-forms to this last background and requiring that the Killing spinors remain unchanged [38, 39]. We will explicitly verify that this requirement allows to determine the metric and forms of this solution.

B.1 Killing spinors of the deformed conifold

We will start by introducing the metric of the deformed conifold. With this purpose, let us define the following set of one-forms

$$
\begin{aligned}
g^1 &= -\frac{1}{\sqrt{2}}\,(\sigma^2 - w^2)\,, \qquad g^2 = \frac{1}{\sqrt{2}}\,(\sigma^1 - w^1)\,, \qquad g^3 = -\frac{1}{\sqrt{2}}\,(\sigma^2 + w^2)\,,\\
g^4 &= \frac{1}{\sqrt{2}}\,(\sigma^1 + w^1)\,, \qquad g^5 = \sigma^3 + w^3\,,
\end{aligned}
\tag{B.3}
$$

where the σ^i have been defined in eq. (2.9) and the w^j are the one-forms displayed in eq. (2.10). The metric of the deformed conifold can be written as [24]:

$$
\begin{aligned}
ds_6^2 = {}& \frac{1}{2}\,\mu^{\frac{4}{3}}\,K(\tau)\,\times\\
&\times\Bigg[\frac{1}{3K(\tau)^3}\,\big(d\tau^2+(g^5)^2\big)+\cosh^2\Big(\frac{\tau}{2}\Big)\,\big((g^3)^2+(g^4)^2\big)+\sinh^2\Big(\frac{\tau}{2}\Big)\,\big((g^1)^2+(g^2)^2\big)\Bigg]\,,
\end{aligned}
\tag{B.4}
$$

where τ is a radial coordinate, μ is the deformation parameter of the conifold and the function $K(\tau)$ is given by:

$$
K(\tau) \equiv \frac{(\sinh 2\tau - 2\tau\,)^{1/3}}{2^{1/3}\,\sinh\tau}\,. \tag{B.5}
$$

Let us now consider the following ten-dimensional metric:

$$
ds_{10}^2 = dx_{1,3}^2 + ds_6^2 = -(dx^0)^2 + \cdots + (dx^3)^2 + ds_6^2\,, \tag{B.6}
$$

where ds_6^2 is the metric of (B.4). This metric determines a Ricci flat geometry which solves the equations of motion of supergravity without forms and preserves eight supersymmetries. As shown in ref. [40], this solution can be naturally obtained by uplifting a domain wall in gauged eight-dimensional supergravity. Actually, this eight-dimensional origin provides us with the insight to find the frame basis in which the Killing spinors become (almost) constant. To illustrate this fact, let us rewrite the conifold metric (B.4) as it is obtained in ref. [40], namely:

$$
\begin{aligned}
ds_6^2 = {}& \frac{1}{2}\,\mu^{\frac{4}{3}}\,K(\tau)\Bigg[\frac{1}{3K(\tau)^3}\,\big(d\tau^2+(w^3+\sigma^3)^2\big)+\frac{\sinh^2\tau}{2\cosh\tau}\,\big((\sigma^1)^2+(\sigma^2)^2\big)+\\
&+\frac{\cosh\tau}{2}\Bigg[\Big(w^1+\frac{\sigma^1}{\cosh\tau}\Big)^2+\Big(w^2+\frac{\sigma^2}{\cosh\tau}\Big)^2\Bigg]\Bigg]\,.
\end{aligned}
\tag{B.7}
$$

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We now choose the following frame one-forms for the ten-dimensional metric (B.6):

$$\begin{aligned}
e^{x^i} &= dx^i\,, \qquad (i=0,1,2,3)\,, \qquad e^{\tau} = \frac{\mu^{2/3}}{\sqrt{6}\,K(\tau)}\,d\tau\,,\\
e^{i} &= \frac{\mu^{2/3}\sqrt{K(\tau)}}{2}\,\frac{\sinh\tau}{\sqrt{\cosh\tau}}\,\sigma^i\,, \qquad (i=1,2)\,,\\
e^{\hat{i}} &= \frac{\mu^{2/3}\sqrt{K(\tau)}}{2}\,\sqrt{\cosh\tau}\,\Big(w^i+\frac{\sigma^i}{\cosh\tau}\Big)\,, \qquad (i=1,2)\,,\\
e^{\hat{3}} &= \frac{\mu^{2/3}}{\sqrt{6}\,K(\tau)}\,(w^3+\sigma^3)\,.
\end{aligned} \tag{B.8}$$

By computing the different components of the spin connection in the basis (B.8), and substituting the results on the right-hand side of eq. (B.1), one can check that the Killing spinors are independent of the angular coordinates of the conifold and must satisfy the following projections:

$$\begin{aligned}
\Gamma_{12}\,\epsilon &= -\Gamma_{\hat{1}\hat{2}}\,\epsilon\,,\\
\Gamma_{\tau\hat{1}\hat{2}\hat{3}}\,\epsilon &= -(\cos\alpha+\sin\alpha\,\Gamma_{\hat{1}1}\,)\,\epsilon\,,
\end{aligned} \tag{B.9}$$

with α being [40]:

$$\cos\alpha = \frac{\sinh\tau}{\cosh\tau}\,, \qquad \sin\alpha = -\frac{1}{\cosh\tau}\,. \tag{B.10}$$

Let us solve eq. (B.9) by representing ϵ as:

$$\epsilon = e^{-\frac{\alpha}{2}\Gamma_{\hat{1}1}}\,\eta\,, \tag{B.11}$$

where η is a spinor satisfying the projections

$$\Gamma_{\tau\hat{1}\hat{2}\hat{3}}\,\eta = -\eta\,, \qquad \Gamma_{12}\,\eta = -\Gamma_{\hat{1}\hat{2}}\,\eta\,. \tag{B.12}$$

By substituting the representation (B.11) on the supergravity variations (B.1) one readily proves that η is constant. Thus, the only dependence of ϵ on the coordinates comes from the angle α, which is only a function of the radial coordinate τ and is such that $\alpha\to 0$ in the ultraviolet region $\tau\to\infty$ (see eq. (B.10)). Therefore, the spinor ϵ at a finite value of τ can be reconstructed from its asymptotic value η by means of a τ-dependent rotation with angle α. Notice that the algebraic conditions (B.12) determine eight spinors η, as previously claimed.

B.2 D3-branes on the deformed conifold

Let us now add D3-branes to the deformed conifold solution. This amounts to taking a metric of the type

$$ds^2_{10} = h^{-1/2}\,dx^2_{1,3} + h^{1/2}\,ds^2_6\,, \tag{B.13}$$

where $h(\tau)$ is a warp factor, and a Ramond-Ramond five-form of the form:

$$g_s\,F^{(5)} = d^4x\,\wedge dh^{-1} + \text{Hodge dual}\,. \tag{B.14}$$

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To determine the Killing spinors for this metric, we will use a frame such as the one in (B.8), which now includes the corresponding powers of the warp factor h. By computing the contributions of the warp factor and $F^{(5)}$ to the gravitino variation in (B.1), one readily realizes that, in addition to (B.9), the Killing spinors ϵ for this background satisfy an additional projection, namely:

$$\Gamma_{x^0x^1x^2x^3}\,\epsilon = -i\epsilon\,. \tag{B.15}$$

Moreover, since the type IIB spinors ϵ have fixed ten-dimensional chirality

$$\Gamma_{x^0x^1x^2x^3}\,\Gamma_{\tau 12\hat{1}\hat{2}\hat{3}}\,\epsilon = -\epsilon\,, \tag{B.16}$$

we have now

$$\Gamma_{\tau\hat{3}}\,\epsilon = -i\epsilon\,, \tag{B.17}$$

where we have used (B.15) and the first equation in (B.9). By combining this last equation with the relations in (B.9) we obtain

$$\Gamma_{12}\,\epsilon = -\Gamma_{\hat{1}\hat{2}}\,\epsilon = i\,(\cos\alpha + \sin\alpha\,\Gamma_{\hat{1}1}\,)\,\epsilon\,. \tag{B.18}$$

The Killing spinors in this case can be represented as:

$$\epsilon = h^{-\frac{1}{8}}\,e^{-\frac{\alpha}{2}\Gamma_{\hat{1}1}}\,\eta\,, \tag{B.19}$$

where η is a constant spinor that satisfies the projections (B.12), together with

$$\Gamma_{x^0x^1x^2x^3}\,\eta = -i\eta\,. \tag{B.20}$$

Notice that the algebraic conditions (B.12) and (B.20) determine four spinors.

B.3 Klebanov-Strassler solutions

Let us now add three-form gauge fields H and $F^{(3)}$ to the solution of section B.2. The metric and five-form will still be given by eqs. (B.13) and (B.14), while the three-forms will be parametrized by means of the ansatz of ref. [23], in which $F^{(3)}$ is written as:

$$F^{(3)} = \frac{M\alpha'}{2}\,\Big[\,g^3\wedge g^4\wedge g^5 + d\,(\,F(\tau)\,(\,g^1\wedge g^3 + g^2\wedge g^4\,)\,)\,\Big]\,. \tag{B.21}$$

In eq. (B.21) M is a constant and $F^{(3)}$ is given in terms of a single radial function $F(\tau)$. The three-form H is obtained from a two-form potential B as $H = dB$. The ansatz of ref. [23] for B depends on two functions $f(\tau)$ and $k(\tau)$ as:

$$B = \frac{g_s M\alpha'}{2}\,\Big[\,f(\tau)\,g^1\wedge g^2 + k(\tau)\,g^3\wedge g^4\,\Big]\,. \tag{B.22}$$

To determine the functions F, f and k, we shall impose to the Killing spinors the same projections as in the D3-brane case and we will require that the contributions of the three-forms to the SUSY variation of the dilatino and gravitino vanish by themselves. Thus, the

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addition of the three-forms will not change the Killing spinors. From the condition $\delta\lambda = 0$ we get the following first-order equation

$$2F' + \coth\left(\frac{\tau}{2}\right) f' - \tanh\left(\frac{\tau}{2}\right) k' + 2\coth\tau\, F + f - k = \tanh\left(\frac{\tau}{2}\right). \tag{B.23}$$

Moreover, from $\delta\psi_{\hat{3}} = 0$ we get the differential equation

$$F' = \frac{1}{2}\coth\left(\frac{\tau}{2}\right) f' - \frac{1}{2}\tanh\left(\frac{\tau}{2}\right) k', \tag{B.24}$$

and the following algebraic relation between the functions entering the ansatz

$$2\coth\tau\, F + k - f = \tanh\left(\frac{\tau}{2}\right). \tag{B.25}$$

Plugging eqs. (B.24) and (B.25) in (B.23), one immediately gets

$$F' = \frac{k-f}{2}, \tag{B.26}$$

which is one of the first-order equations found in ref. [23]. By inserting eq. (B.26) into eq. (B.25), we get the following differential equation which only involves F and its first derivative

$$F' + \coth\tau\, F = \frac{1}{2}\tanh\left(\frac{\tau}{2}\right). \tag{B.27}$$

This equation can be integrated easily and the explicit form of F can be obtained (see below). By substituting the value of F' given by eq. (B.27) into eq. (B.24), we get:

$$\coth\left(\frac{\tau}{2}\right) f' - \tanh\left(\frac{\tau}{2}\right) k' = -2\coth\tau\, F + \tanh\left(\frac{\tau}{2}\right). \tag{B.28}$$

More equations are obtained from the SUSY variations of other components of the gravitino. Indeed, from the condition $\delta\psi_1 = 0$ we obtain two other equations

$$f' - k' - 2F + 1 - 2\coth\tau\left[F' + \frac{k-f}{2}\right] = 0, \tag{B.29}$$

$$\coth^2\left(\frac{\tau}{2}\right) f' - \tanh^2\left(\frac{\tau}{2}\right) k' - 2(\coth^2\tau + \mathrm{csch}^2\tau)F + \tanh^2\left(\frac{\tau}{2}\right) - \\ -2\coth\tau\left[F' + \frac{k-f}{2}\right] = 0. \tag{B.30}$$

Notice that, by combining (B.26) and (B.27), we get:

$$2F' + \frac{k-f}{2} = 2F' = \tanh\left(\frac{\tau}{2}\right) - 2\coth\tau\, F. \tag{B.31}$$

Let us now use this result to compute the last term on the left-hand side of eq. (B.29). After some calculation, eq. (B.29) becomes

$$f' - k' = -\left[\coth^2\left(\frac{\tau}{2}\right) + \tanh^2\left(\frac{\tau}{2}\right)\right] F + \tanh^2\left(\frac{\tau}{2}\right). \tag{B.32}$$

By combining eqs. (B.28) and (B.32) one can get f' and k' as functions of F. The result is

$$f' = (1-F)\tanh^2\left(\frac{\tau}{2}\right), \qquad k' = F\coth^2\left(\frac{\tau}{2}\right). \tag{B.33}$$

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Eqs. (B.26) and (B.33) constitute the first-order system of ref. [23]. Apart from this system, we have obtained, in addition, a new differential equation (eq. (B.27)) or, alternatively, the algebraic relation (B.25). It can be checked that eq. (B.30) follows from (B.26), (B.33) and (B.27). Moreover the SUSY variations of the remaining components of the gravitino also cancel as a consequence of these equations.

Eq. (B.27) can be easily integrated by the method of variation of constants. The result is:

$$F = \frac{1}{2}\,\frac{\sinh\tau - \tau}{\sinh\tau} + \frac{A}{\sinh\tau}\,, \tag{B.34}$$

where A is a constant. By requiring regularity of F at $\tau = 0$ we fix the constant A to the value $A = 0$. It is then immediate to integrate the first-order equations for f and k. The result is the same as in ref. [23], namely:

$$\begin{aligned} F &= \frac{1}{2}\,\frac{\sinh\tau - \tau}{\sinh\tau}\,, \\ f &= \frac{1}{2}\,\frac{\tau\coth\tau - 1}{\sinh\tau}\,(\cosh\tau - 1)\,, \\ k &= \frac{1}{2}\,\frac{\tau\coth\tau - 1}{\sinh\tau}\,(\cosh\tau + 1)\,. \end{aligned} \tag{B.35}$$

Thus, the requirement of preserving the same supersymmetries as in the solution corresponding to a D3-brane at the tip of a deformed conifold fixes the values of the three-forms to those found in ref. [23]. Therefore, we conclude that the Killing spinors of the Klebanov-Strassler background are given by eq. (B.19), with η satisfying the projections (B.12) and (B.20).

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